UNIT -5

• Dimensional Analysis
  – Fundamental Dimensions
  – Dimensional Homogeneity
  – Method of analysis
    • Rayleigh Method
    • Buckingham pi theorem Method

• Model Analysis
  – Dimensionless parameters
  – Similitude and model studies
  – Distorted Models
## DIMENSIONS

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Rayleigh Method
Problem 12.3  Find an expression for the drag force on smooth sphere of diameter $D$, moving with a uniform velocity $V$ in a fluid of density $\rho$ and dynamic viscosity $\mu$.

Solution.  Drag force $F$ is a function of

(i) Diameter, $D$
(ii) Velocity, $V$
(iii) Density, $\rho$
(iv) Viscosity, $\mu$

\[ F = KD^a \cdot V^b \cdot \rho^c \cdot \mu^d \]

where $K$ is non-dimensional factor.

Substituting the dimensions on both sides,

\[ MLT^{-2} = KL^a \cdot (LT^{-1})^b \cdot (ML^{-3})^c \cdot (ML^{-1}T^{-1})^d \]

Equating the powers of $M, L$ and $T$ on both sides,

Power of $M$,
\[ 1 = c + d \]

Power of $L$,
\[ 1 = a + b - 3c - d \]

Power of $T$,
\[ -2 = -b - d. \]

There are four unknowns ($a, b, c, d$) but equations are three. Hence it is not possible to find the exact values of $a, b, c$ and $d$. But three of them can be expressed in terms of fourth variable which is most important. Here viscosity is having a vital role and hence $a, b, c$ are expressed in terms of $d$ which is the power to viscosity.

\[ c = 1 - d \]
\[ b = 2 - d \]
\[ a = 1 - b + 3c + d = 1 - 2 + d + 3(1 - d) + d \]

Substituting these values of $a, b$ and $c$ in (i), we get

\[ F = KD^{2-d} \cdot V^{2-d} \cdot \rho^{1-d} \cdot \mu^d \]

\[ = KD^2 V^2 \rho (D^{-d} \cdot V^{-d} \cdot \rho^{-d} \cdot \mu^d) = K \rho D^2 V^2 \left( \frac{\mu}{\rho V D} \right)^d \]

\[ = K \rho D^2 V^2 \left( \frac{\mu}{\rho V D} \right) \cdot \text{Ans.} \]
Buckingham $\Pi$ Theorem
Definition for Pi theorem

The difficulty is overcome by using Buckingham's \( \pi \)-theorem, which states, "If there are \( n \) variables (independent and dependent variables) in a physical phenomenon and if these variables contain \( m \) fundamental dimensions \( (M, L, T) \), then the variables are arranged into \( (n - m) \) dimensionless terms. Each term is called \( \pi \)-term".

Let \( X_1, X_2, X_3, \ldots, X_n \) are the variables involved in a physical problem. Let \( X_1 \) be the dependent variable and \( X_2, X_3, \ldots, X_n \) are the independent variables on which \( X_1 \) depends. Then \( X_1 \) is a function of \( X_2, X_3, \ldots, X_n \) and mathematically it is expressed as

\[
X_1 = f(X_2, X_3, \ldots, X_n)
\]

Equation (12.1) can also be written as

\[
f_1(X_1, X_2, X_3, \ldots, X_n) = 0.
\]
Problem 12.13  The frictional torque $T$ of a disc of diameter $D$ rotating at a speed $N$ in a fluid of viscosity $\mu$ and density $\rho$ in a turbulent flow is given by

$$T = D^5 N^2 \rho \phi \left( \frac{\mu}{D^2 N \rho} \right).$$

Prove this by the method of dimensions.

Solution.  Given:  

$$T = f(D, N, \mu, \rho) \text{ or } f_1(T, D, N, \mu, \rho) = 0 \quad ... (i)$$

:.  Total number of variables, $n = 5$

Dimensions of each variable are expressed as

$$T = ML^2 T^{-2}, D = L, N = T^{-1}, \mu = ML^{-1} T^{-1}, \rho = ML^{-3}$$

:.  Number of fundamental dimensions, $m = 3$

Number of $\pi$-terms

$$= n - m = 5 - 3 = 2$$

Hence equation (i) can be written as $f_1(\pi_1, \pi_2) = 0 \quad ... (ii)$

Each $\pi$-term contains $m + 1$ variable, i.e., $3 + 1 = 4$ variables. Three variables are repeating variables. Choosing $D, N, \rho$ as repeating variables, the $\pi$-terms are

$$\pi_1 = D^{a_1} N^{b_1} \rho^{c_1} T$$

$$\pi_2 = D^{a_2} N^{b_2} \rho^{c_2} \mu$$
Step 2

Dimensional Analysis of $\pi_1$

$$\pi_1 = D^{a_1} \cdot N^{b_1} \cdot \rho^{c_1} \cdot T$$

Substituting dimensions on both sides,

$$M^0L^0T^0 = L^{a_1} \cdot (T^{-1})^{b_1} \cdot (ML^{-3})^{c_1} \cdot ML^2T^{-2}.$$  

Equating the powers of $M$, $L$, $T$ on both sides,

Power of $M$,

$$0 = c_1 + 1$$

$\therefore$ $c_1 = -1$

Power of $L$,

$$0 = a_1 - 3c_1 + 2$$

$\therefore$ $a_1 = 3c_1 - 2 = -3 - 2 = -5$

Power of $T$,

$$0 = -b_1 - 2$$

$\therefore$ $b_1 = -2$

Substituting the values of $a_1$, $b_1$, $c_1$ in $\pi$,

$$\pi_1 = D^{-5} \cdot N^{-2} \cdot \rho^{-1} \cdot T = \frac{T}{D^5N^2\rho}.$$

• Step-3

Dimensional Analysis of $\Pi_2 = D^{a_2} \cdot N^{b_2} \cdot \rho^{c_2} \cdot \mu$

Substituting dimensions on both sides,

$$M^0L^0T^0 = L^{a_2} \cdot (T^{-1})^{b_2} \cdot (ML^{-3})^{c_2} \cdot ML^{-1}T^{-1}.$$  

Equating the powers of $M$, $L$, $T$ on both sides,

Power of $M$,

$$0 = c_2 + 1$$

$\therefore$ $c_2 = -1$

Power of $L$,

$$0 = a_2 - 3c_2 - 1$$

$\therefore$ $a_2 = 3c_2 + 1 = -3 + 1 = -2$

Power of $T$,

$$0 = -b_2 - 1$$

$\therefore$ $b_2 = -1$

Substituting the values of $a_2$, $b_2$ and $c_2$ in $\pi_2$,

$$\pi_2 = D^{-2}N^{-1}\rho^{-1} \cdot \mu = \frac{\mu}{D^2N\rho}.$$
Substituting the values of $\pi_1$ and $\pi_2$ in equation (ii),

$$f_1 \left( \frac{T}{D^5 N^2 \rho}, \frac{\mu}{D^2 N \rho} \right) = 0 \quad \text{or} \quad f_1 \left( \frac{T}{D^5 N^2 \rho}, \frac{\mu}{D^2 N \rho} \right) = \phi \left( \frac{\mu}{D^2 N \rho} \right)$$

or

$$T = D^5 N^2 \rho \phi \left[ \frac{\mu}{D^2 N \rho} \right]. \text{ Ans.}$$
Problem 2

The pressure difference $\Delta p$ in a pipe of diameter $D$ and length $l$ due to viscous flow depends on the velocity $V$, $\mu$ viscosity and $\rho$ density. Using Buckingham’s theorem, obtain an expression for $\Delta p$

Now $\Delta p$ is a function of $D$, $l$, $V$, $\mu$, $\rho$ or $\Delta p = f(D, l, V, \mu, \rho)$

or $f_1(\Delta p, D, l, V, \mu, \rho) = 0$

Total number of variables, $n = 6$

Number of fundamental dimension, $m = 3$

Number of $\pi$-terms $= n - m = 6 - 3 = 3$

Hence equation (i) is written as $f_1(\pi_1, \pi_2, \pi_3) = 0$

\[\text{(ii)}\]

Each $\pi$-term contains $m + 1$ variables, i.e., $3 + 1 = 4$ variable. Out of four variables, three are repeating variables.

Choosing $D$, $V$, $\mu$ as repeating variables, we have $\pi$-terms as

$\pi_1 = D^{a_1} \cdot V^{b_1} \cdot \mu^{c_1} \cdot \Delta p$

$\pi_2 = D^{a_2} \cdot V^{b_2} \cdot \mu^{c_2} \cdot l$

$\pi_3 = D^{a_3} \cdot V^{b_3} \cdot \mu^{c_3} \cdot \rho$
First $\pi$-term

\[ \pi_1 = D^{a_1} \cdot V^{b_1} \cdot \mu^{c_1} \cdot \Delta p \]

Substituting the dimensions on both sides,

\[ M^0 L^0 T^0 = L^{a_1} \cdot (LT^{-1})^{b_1} \cdot (ML^{-1} T^{-1})^{c_1} \cdot ML^{-1} T^{-2} \]

Equating the powers of $M$, $L$, $T$ on both sides,

- Power of $M$, \( 0 = c_1 + 1, \quad \therefore c_1 = -1 \)
- Power of $L$, \( 0 = a_1 + b_1 - c_1 - 1, \quad \therefore a_1 = -b_1 + c_1 + 1 = 1 - 1 + 1 = 1 \)
- Power of $T$, \( 0 = -b_1 - c_1 - 2, \quad \therefore b_1 = -c_1 - 2 = 1 - 2 = -1 \)

Substituting the values of $a_1$, $b_1$ and $c_1$ in $\pi_1$,

\[ \pi_1 = D^1 \cdot V^{-1} \cdot \mu^{-1} \cdot \Delta p = \frac{D\Delta p}{\mu V} \]

Second $\pi$-term

\[ \pi_2 = D^{a_2} \cdot V^{b_2} \cdot \mu^{c_2} \cdot l \]

Substituting the dimensions on both sides,

\[ M^0 L^0 T^0 = L^{a_2} \cdot (LT^{-1})^{b_2} \cdot (ML^{-1} T^{-1})^{c_2} \cdot L \]

Equating the powers of $M$, $L$, $T$ on both sides

- Power of $M$, \( 0 = c_2, \quad \therefore c_2 = 0 \)
- Power of $L$, \( 0 = a_2 + b_2 - c_2 + 1, \quad \therefore a_2 = -b_2 + c_2 - 1 = -1 \)
- Power of $T$, \( 0 = -b_2 - c_2, \quad \therefore b_2 = -c_2 = 0 \)

Substituting the values of $a_2$, $b_2$ and $c_2$ in $\pi_2$,
Third π-term \[ \pi_3 = D^{a_3} \cdot V^{b_3} \cdot \mu^{c_3} \cdot \rho \]

Substituting the dimension on both sides,
\[ M^0L^0T^0 = L^{a_3} \cdot (LT^{-1})^{b_3} \cdot (ML^{-1}T^{-1})^{c_3} \cdot ML^{-3}. \]

Equating the powers of \( M, L, T \) on both sides

Power of \( M, \) \[ 0 = c_3 + 1, \quad \therefore c_3 = -1 \]

Power of \( L, \) \[ 0 = a_3 + b_3 - c_3 - 3, \quad \therefore a_3 = -b_3 + c_3 + 3 = -1 - 1 + 3 = 1 \]

Power of \( T, \) \[ 0 = -b_3 - c_3, \quad \therefore b_3 = -c_3 = -(-1) = 1 \]

Substituting the values of \( a_3, b_3 \) and \( c_3 \) in \( \pi_3, \)
\[ \pi_3 = D^1 \cdot V^1 \cdot \mu^{-1} \cdot \rho = \frac{\rho DV}{\mu}. \]

Substituting the values of \( \pi_1, \pi_2 \) and \( \pi_3 \) in equation (ii),
\[ f_1 \left( \frac{D\Delta p}{\mu V}, \frac{l}{D}, \frac{\rho DV}{\mu} \right) = 0 \quad \text{or} \quad \frac{D\Delta p}{\mu V} = \phi \left[ \frac{l}{D}, \frac{\rho DV}{\mu} \right] \quad \text{or} \quad \Delta p = \frac{\mu V}{D} \phi \left[ \frac{l}{D}, \frac{\rho DV}{\mu} \right] \]
**Problem 12.9** Using Buckingham's \( \pi \)-theorem, show that the velocity through a circular orifice is given by

\[
V = \sqrt{2gH} \phi \left[ \frac{D}{H}, \frac{\mu}{\rho VH} \right],
\]

where \( H \) is the head causing flow, \( D \) is the diameter of the orifice, \( \mu \) is co-efficient of viscosity, \( \rho \) is the mass density and \( g \) is the acceleration due to gravity.

**Solution.** Given:

- \( V \) is a function of \( H, D, \mu, \rho \) and \( g \)
- \( V = f(H, D, \mu, \rho, g) \) or \( f_1(V, H, D, \mu, \rho, g) = 0 \)
- Total number of variable, \( n = 6 \)

Writing dimension of each variable, we have

\[
V = LT^{-1}, \quad H = L, \quad D = L, \quad \mu = ML^{-1}T^{-1}, \quad \rho = ML^{-3}, \quad g = LT^{-2}.
\]

Thus number of fundamental dimensions, \( m = 3 \)
- Number of \( \pi \)-terms \( = n - m = 6 - 3 = 3 \).
- Equation (i) can be written as \( f_1(\pi_1, \pi_2, \pi_3) = 0 \) \( \ldots (ii) \)
- Each \( \pi \)-term contains \( m + 1 \) variables, where \( m = 3 \) and is also equal to repeating variables. Here \( V \) is a dependent variable and hence should not be selected as repeating variable. Choosing \( H, g, \rho \) as repeating variable, we get three \( \pi \)-terms as

\[
\begin{align*}
\pi_1 &= H^{a_1} \cdot g^{b_1} \cdot \rho^{c_1} \cdot V \\
\pi_2 &= H^{a_2} \cdot g^{b_2} \cdot \rho^{c_2} \cdot D \\
\pi_3 &= H^{a_3} \cdot g^{b_3} \cdot \rho^{c_3} \cdot \mu
\end{align*}
\]
First $\pi$-term

$$\pi_1 = H^{a_1} \cdot g^{b_1} \cdot \rho^{c_1} \cdot V$$

Substituting dimensions on both sides

$$M^a L^b T^c = (LT^{-2})^{a_1} \cdot (MT^{-3})^{b_1} \cdot (LT^{-1})^{c_1}$$

Equating the powers of $M, L, T$ on both sides,

Power of $M$,

$$0 = c, \quad \therefore c_1 = 0$$

Power of $L$,

$$0 = a_1 + b_1 - 3c_1 + 1, \quad \therefore a_1 = -b_1 + 3c_1 - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

Power of $T$,

$$0 = -2b_1 - 1, \quad \therefore b_1 = -\frac{1}{2}$$

Substituting the values of $a_1, b_1$ and $c_1$ in $\pi_1$,

$$\pi_1 = H^{-\frac{1}{2}} \cdot g^{-\frac{1}{2}} \cdot \rho^0 \cdot V = \frac{V}{\sqrt{gH}}.$$
Second $\pi$-term

$$\pi_2 = H^{a_2} \cdot g^{b_2} \cdot \rho^{c_2} \cdot D$$

Substituting the dimensions on both sides,

$$M^0 L^0 T^0 = L^{a_2} \cdot (LT^{-2})^{b_2} \cdot (ML^{-3})^{c_2} \cdot L$$

Equating the powers of $M$, $L$, $T$,

Power of $M$, $0 = c_2$ \quad \therefore \quad c_2 = 0$

Power of $L$, $0 = a_2 + b_2 - 3c_2 + 1$, $a_2 = -b_2 + 3c_2 - 1 = -1$

Power of $T$, $0 = -2b_2$, \quad \therefore \quad b_2 = 0$

Substituting the values of $a_2$, $b_2$, $c_2$ in $\pi_2$,

$$\pi_2 = H^{-1} \cdot g^0 \rho^0 \cdot D = \frac{D}{H}.$$
Third \( \pi \)-term 
\[ \pi_3 = H^{a_3} \cdot g^{b_3} \cdot \rho^{c_3} \cdot \mu \]
Substituting the dimensions on both sides
\[ M^0 L^b T^c = L^{a_3} \cdot (L T^{-2})^{b_3} \cdot (ML^{-3})^{c_3} \cdot ML^{-1} T^{-1} \]
Equating the powers of \( M, L, T \) on both sides
Power of \( M \),
\[ 0 = c_3 + 1, \quad \therefore \quad c_3 = -1 \]
Power of \( L \),
\[ 0 = a_3 + b_3 - 3c_3 - 1, \quad \therefore \quad a_3 = -b_3 + 3c_3 + 1 = \frac{1}{2} - 3 + 1 = -\frac{3}{2} \]
Power of \( T \),
\[ 0 = -2b_3 - 1, \quad \therefore \quad b_3 = -\frac{1}{2} \]
Substituting the values of \( a_3, b_3 \) and \( c_3 \) in \( \pi_3 \),
\[ \pi_3 = H^{-3/2} \cdot g^{-1/2} \cdot \rho^{-1} \cdot \mu = \frac{\mu}{H^{3/2} \rho \sqrt{g}} \]
\[ = \frac{\mu}{H \rho \sqrt{gH}} = \frac{\mu V}{H \rho V \sqrt{gH}} \quad \text{[Multiply and Divide by 1]} \]
\[ = \frac{\mu}{H \rho V} \cdot \pi_1 \]
\[ \therefore \quad \frac{V}{\sqrt{gH}} = \pi_1 \]
Substituting the values of \( \pi_1 \) in equation (ii),
\[ f_1 \left( \frac{V}{\sqrt{gH}}, \frac{D}{H}, \pi_1, \frac{\mu}{H \rho V} \right) = 0 \text{ or } \frac{V}{\sqrt{gH}} = \phi \left[ \frac{D}{H}, \pi_1, \frac{\mu}{H \rho V} \right] \]
UNIT -5

• Dimensional Analysis

• Model Analysis
  – Dimensionless parameters
    • Reynolds No, Froude No, Math No, Weber No & Euler No
  – Similitude and model studies
    • Geometric Similarity
    • Kinematic Similarity
    • Dynamic Similarity
  – Distorted Models
    • Scale ratios are different in vertical and horizontal directions
Model & Dimensional Analysis

Dam, Reservoir, River

If model in Q & g, Dam Q = ?
If Dam Pr. = 1000 kw, Model Dam Pr. = ?
If River Q = g, Then model River Q = ?

Zone I - If gravity flow Predominant Model Sea wave

Zone II - Pipe line flow oil flow, Reynolds Ship rub with water

Zone III - Aeroplane high speed, Man Model law (water hammer)

Zone IV - Euler's - Pipe line with high Pr.

Zone V - Capillary - Wfeber.
Basic Terms

• Model
  – A small scale replica of the actual structure or machine.

• Prototype
  – The actual structure itself.

• Model Analysis
  – An experimental method of findings solutions to complex flow problems.
Advantages of model analysis (related to civil engineering)

- **Performance** of the hydraulic structure can be predicted.
- With the help of dimensional analysis, a relationship between the variables influencing flow problems in terms of *dimensionless numbers* is obtained. This helps in conducting the test on the model.
- The **most economical and safe design** can be found out.
Similitude or Similarity

• Geometric Similarity

\[ \frac{L_p}{L_m} = \frac{b_p}{b_m} = \frac{D_p}{D_m} = L_r \]

Here \( p \) represents prototype

\( m \) represents model

\( r \) represents ratio (scale ratio)

Area - \( L_r^2 \)

Volume - \( L_r^3 \)

• Kinematic Similarity

\[ \frac{V_{p1}}{V_{m1}} = \frac{v_{p2}}{v_{m2}} = v_r \]

• Dynamic Similarity

\[ \frac{(F_i)_p}{(F_i)_m} = \frac{(F_v)_p}{(F_v)_m} = \frac{(F_g)_p}{(F_g)_m} = ... = F_r, \text{ where } F_r \text{ is the force ratio} \]

i - inertia Force

v - viscous Force

g – gravity force
Dimensionless Number

• Reynolds Number
• Froude Number
• Mach Number
• Euler Number
• Weber Number
1. **Reynold’s number**: It is defined as the ratio of inertia force of a flowing fluid and the viscous force of the fluid. The expression for Reynold’s number is obtained as

\[
\text{Inertia force } (F_i) = \text{Mass} \times \text{Acceleration of flowing fluid}
\]

\[
\text{Inertia force } (F_i) = \rho \times \text{Volume} \times \frac{\text{Velocity}}{\text{Time}} = \rho \times \frac{\text{Volume}}{\text{Time}} \times \text{Velocity}
\]

\[
= \rho \times AV \times V \quad \text{Volume per sec} = \text{Area} \times \text{Velocity} = A
\]

\[
= AV \times V
\]

\[
\text{Inertia force } (F_i) = \rho A V^2
\]

\[
\text{Viscous force } (F_v) = \text{Shear stress} \times \text{Area}
\]

\[
\text{Viscous force } (F_v) = \tau \times A
\]

\[
= \left( \frac{\mu \, du}{dy} \right) \times A = \mu \frac{V}{L} \times A
\]

\[
\frac{du}{dy} = \frac{V}{L}
\]

By definition, Reynold’s number

\[
R_e = \frac{F_i}{F_v} = \frac{\rho A V^2}{\mu \frac{V}{L} \times A} = \frac{\rho V L}{\mu} = \frac{V \times L}{\mu / \rho} = \frac{V \times L}{\nu} \quad \mu / \rho = \nu
\]

In case of pipe flow, the linear dimension \(L\) is taken as diameter, \(d\). Hence Reynold’s number for pipe flow,

\[
R_e = \frac{V \times d}{v} \quad \text{or} \quad \frac{\rho V d}{\mu}
\]
2. Froude's Number (Fe): The Froud's Number is defined as the square root of the ratio of inertia force of a flowing fluid to the gravitational force. Mathematically, it is expressed as

\[ F_e = \sqrt{\frac{F_i}{F_g}} \]

Inertia force \((F_i) = \rho AV^2\)

\[ F_g = \text{Force due to gravity} = \text{Mass} \times \text{Acceleration due to gravity} = \rho X L^3 \times g \]

\[ = \rho X L^2 \times L \times g = \rho X A \times L \times g \]

\[ F_e = \sqrt{\frac{F_i}{F_g}} = \sqrt{\frac{\rho AV^2}{\rho ALg}} = \sqrt{\frac{V^2}{Lg}} = \frac{V}{\sqrt{Lg}} \]
Scale Ratio

(a) Scale ratio for time

\[
\text{Time} = \frac{\text{Length}}{\text{Velocity}}
\]

then ratio of time for prototype and model is

\[
T_r = \frac{T_p}{T_m} = \frac{\frac{L_p}{V_p}}{\frac{L_m}{V_m}} = \frac{L_p}{L_m} \times \frac{V_m}{V_p} = L_r \times \frac{1}{\sqrt{L_r}} = \sqrt{L_r}
\]

\[
\frac{V_p}{V_m} = \sqrt{L_r}
\]

(b) Scale ratio for acceleration

\[
\text{Acceleration} = \frac{V}{T}
\]

\[
a_r = \frac{a_p}{a_m} = \frac{\frac{V}{T_p}}{\frac{V}{T_m}} = \frac{V_p}{V_m} \times \frac{T_m}{T_p} = \frac{V_p}{V_m} \times \frac{T_m}{T_p} = \sqrt{L_r} \times \frac{1}{\sqrt{L_r}} = 1
\]

\[
\frac{V_p}{V_m} = \sqrt{L_r}, \quad \frac{T_p}{T_m} = \sqrt{L_r}
\]
(c) Scale ratio for discharge

\[ Q = A \times V = L^2 \times \frac{L}{T} = \frac{L^3}{T} \]

\[ Q_r = \frac{Q_p}{Q_m} = \left( \frac{L^3}{T} \right)_p \left( \frac{L^3}{T} \right)_m = \left( \frac{L_p}{L_m} \right)^3 \times \left( \frac{T_m}{T_p} \right) = \times \frac{1}{\sqrt{L_r}} = L_r^{2.5} \]

(d) Scale ratio for force

\[ \text{Force} = \text{Mass} \times \text{Acceleration} = \rho L^3 \times \frac{V}{T} = \rho L^2 \times \frac{L}{T} \cdot V = \rho L^2 V^2 \]

Ratio for force,

\[ F_r = \frac{F_p}{F_m} = \frac{\rho_p L^2 V^2}{\rho_m L^2_m V^2_m} = \frac{\rho_p}{\rho_m} \times \left( \frac{L_p}{L_m} \right)^2 \times \left( \frac{V_p}{V_m} \right)^2 \]

If the fluid used in model and prototype is same, then

\[ \frac{\rho_p}{\rho_m} = 1 \quad \text{(or)} \quad \rho_p = \rho_m L_r^3 \]

\[ F_r = \left( \frac{L_p}{L_m} \right)^2 \times \left( \frac{V_p}{V_m} \right)^2 = L_r^2 \times (\sqrt{L_r})^2 = L_r^2 \cdot L_r = L_r^3 \]
(e) Scale ratio for pressure intensity

\[ p = \frac{\text{Force}}{\text{Area}} = \frac{\rho L^2 V^2}{L^2} = \rho V^2 \]

Pressure ratio, \[ p_r = \frac{p_p}{p_m} = \frac{\rho_p V_p^2}{\rho_m V_m^2} \]

If fluid is same, then \[ \rho_p = \rho_m \]

\[ p_r = \frac{V_p^2}{V_m^2} = \left(\frac{V_p}{V_m}\right)^2 = L_r \]

(f) Scale ratio for work, energy, torque, moment etc.

Torque = Force X Distance = F X L

Torque ratio, \[ T_r^* = \frac{T_p^*}{T_m^*} = \frac{(F \times L)p}{(F \times L)m} = F_r \times L_r = L_1^3 \times L_r = L_r^4 \]
(g) Scale ratio for power

Power = Work per unit time

Power = \( \frac{F \times L}{T} \)

Power ratio, \( P_r \) = \( \frac{p_p}{p_m} = \frac{\frac{F_p \times L_p}{T_p}}{\frac{F_m \times L_m}{T_m}} = \frac{F_p}{F_m} \times \frac{L_p}{L_m} \times \frac{1}{\frac{T_p}{T_m}} \)

\( P_r = F_r \cdot L_r \cdot \frac{1}{T_r} = L_r^3 \times L_r \times \frac{1}{\sqrt{L_r}} = L_r^{3.5} \)
3. **Euler’s number (Eu):** It is defined as the square root of the ratio of inertia force of a flowing fluid to the surface tension force. Mathematically, it is expressed as

\[
\text{Euler's number } (E_u) = \sqrt{\frac{F_i}{F_p}}
\]

\[F_p = \text{Intensity of pressure } \times \text{Area } = \rho \times A\]

Inertia force \((F_i) = \rho \times A \times V^2\)

\[
E_u = \sqrt{\frac{F_i}{F_p}} = \sqrt{\frac{\rho AV^2}{p \times A}} = \sqrt{\frac{V^2}{\rho / \rho}} = \frac{V}{\sqrt{\rho / \rho}}
\]

4. **Weber’s number (We):** It is defined as the square root of the ratio of inertia force of a flowing fluid to the surface tension force. Mathematically, it is expressed as

\[
\text{Weber's number } (W_e) = \sqrt{\frac{F_i}{F_s}}
\]

Inertia force \((F_i) = \rho \times A \times V^2\)

\[F_s = \text{Surface tension force } = \text{Surface tension per unit length } \times \text{Length } = \sigma \times L\]

\[
W_e = \sqrt{\frac{F_i}{F_s}} = \sqrt{\frac{\rho AV^2}{\sigma \times L}} = \sqrt{\frac{\rho \times L^2 V^2}{\sigma \times L}}
\]
5. **Mach number** ($M$): Mach number is defined as the square root of the ratio of inertia force of a flowing fluid to the elastic force. Mathematically, it is expressed as

\[
M = \sqrt{\frac{\text{Inertia force}}{\text{Elastic force}}} = \sqrt{\frac{F_i}{F_e}}
\]

\[
F_i = \rho A V^2
\]

\[
F_e = \text{Elastic force} = \text{Elastic stress} \times \text{Area} = K \times A = K \times L^2
\]

\[
M = \sqrt{\frac{\rho A V^2}{K \times L^2}} = \sqrt{\frac{\rho \times L^2 V^2}{K \times L^2}} = \sqrt{\frac{V^2}{K/\rho}} = \frac{V}{\sqrt{K/\rho}}
\]

\[
\sqrt{\frac{K}{\rho}} = C = \text{Velocity of sound in the fluid}\]

\[
M = \frac{V}{C}
\]
Types of Model

• **Undistorted Models**
  Models that are geometrically similar to their prototypes (i.e., scale ratios for the all directions of linear dimensions in model and prototype are same)

• **Distorted Models**
  Models in which the different scale ratios are used for linear dimensions
  eg river model, harbours model.

**Scale Effect**

If complete similarity does not exist in model and its prototype there will be some discrepancy between the results obtained from model when compared with results in prototype. This effect is called scale effect.
Reynold’s No problem

- Water is flowing through a pipe of diameter 30cm at a velocity of 4m/s. Find the velocity of oil flowing in another pipe of diameter 10cm, if the condition for dynamic similarity is satisfied. The viscosity of water and oil is given by 0.01 poise and 0.025 poise. (sp. Gravity of oil = 0.8).

Pipe 1 - Water
- Diameter $d_1 = 30 \text{ cm} = 0.3 \text{ m}$
- Velocity $v_1 = 4 \text{ m/s}$
- Density $\rho_1 = 1000 \text{ kg/m}^3$
- Viscosity $\mu_1 = 0.001 \text{ N-s/m}^2$

Pipe 2 - Oil
- Diameter $d_2 = 10 \text{ cm} = 0.1 \text{ m}$
- Velocity $v_2 = ? \text{ m/s}$
- Density $\rho_2 = 1000 \times 0.8 \text{ kg/m}^3$
- Viscosity $\mu_2 = 0.0025 \text{ N-s/m}^2$

\[
\frac{\rho_1 \times V_1 \times D_1}{\mu_1} = \frac{\rho_2 \times V_2 \times D_2}{\mu_2}
\]

Find $V_2$

Ans: $V_2 = 35.2 \text{ m/s}$
Problem related to Froude No

• A spillway model is to be built to a geometrically similar scale of 1/50 across a flume of 600 mm width. The prototype is 15m high and maximum head on it is expected to be 1.5m.
  – (i) What height of model and what head on the model to be used
  – (ii) If the flow over the model at a particular head is 12 litres per second, what flow per metre length of the prototype is expected?
  – (iii) If the negative pressure in the model is 200 mm, what is the negative pressure in the prototype.
Scale = (1/50)
1 unit in model = 50 units in prototype
\((L_p/L_m) = L_r = 50\)

- **Model**
  - Width \(B_m = 0.6\)m
  - Height \(H_m = ?\)m
  - Maxi head \(H_{m*} = ?\)m
  - \(Q_m = 12\)lit/sec
  - -ve pr. \(h_m = -0.2\) m

- **Prototype**
  - Width \(B_p = ?\)
  - Height \(H_p = 15\)m
  - Maxi head \(H_{p*} = 1.5\)m
  - \(Q_p = ? \text{ m}^3/\text{sec}\)
  - -ve pr. \(h_p = ?\)m

Width of prototype = 0.6 * 50 = 30 m
Height of model = 15/50 = 0.3m
Maxi. Head of model = 1.5 / 50 = 0.03m = 3 cm
Scale ratio for \(Q_p/Q_m = L_r^{2.5} = 50^{2.5} = 17677.67\)
So discharge in prototype \(Q_p = 12 \times 10^{-3} \times 17677 = 212 \text{ m}^3/\text{s}\)
Discharge per unit width \(q_p = 212 / 30 = 7.07 \text{ m}^3/\text{s}\)
-ve head on prototype = - 0.2 x 50 = - 10m
34. Resistance $R$, to the motion of a completely sub-merged body is given by

$$R = \rho V^2 l^2 \varphi \left( \frac{V}{v} \right)$$

where $\rho$ and $v$ are density and kinematic viscosity of the fluid while $I$ is the length of the body and $V$ is the velocity of flow. If the resistance of a one-eight scale air-ship model when tested in water at 12 m/s is 22 N, what will be the resistance in air of the air-ship at the corresponding speed? Kinematic viscosity of air is 13 times that of water and density of water is 810 times of air.

- **Model - water**
  - $V_m = 12$ m/s
  - $R_m = 22$ N
- **Prototype – Air**
  - $V_p = ?$
  - $R_p = ?$

\[
\begin{align*}
  v_p &= 13 \times v_m \\
  \left( \frac{v_p}{v_m} \right) &= 13 \\
  \rho_m &= 810 \rho_p \\
  \left( \frac{\rho_p}{\rho_m} \right) &= 1/810
\end{align*}
\]
• Step 1 Find $V_p$
Using relation $\left( \frac{V L}{\nu} \right)_p = \left( \frac{V L}{\nu} \right)_m$

• Step 2: Find $R_p$
Using Relation $\left( \frac{R}{\rho V^2 L^2} \right)_p = \left( \frac{R}{\rho V^2 L^2} \right)_m$

Ans:
$V_p = 19.5 \text{ m/s}$
$R_p = 4.59 \text{N}$
Distorted Model Problem

• The discharge through the weir is 1.5 m$^3$/s. Find the discharge through the model of the weir if the horizontal dimension scale is 1/50 and the vertical dimension scale is 1/10.

Ans : $Q_m = 9.48 \times 10^{-4}$ m$^3$/s
Reference
