

ORDINARY DIFFERENTIAL EQUATION:

 $\mathbf{BY}$ 

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**APPLIED MATHEMATICS** 

# Introduction: Differential equation:

An equation involving independent & dependent variables & the derivatives of dependent variables w.r.t independent variables is called a differential equation.

ie. if x is an independent variable, y is the dependent variable then an equation involving, x, y

&  $\frac{dy}{dx}$ ,  $\frac{d^2y}{dx^2}$ , etc is called a differential equation.

Differential equation are of 2 types

(i)**O**. **D**. **E**:

A differential equation which involves derivatives w.r.t a single independent variable is known as O.D.E.

$$Eg: \frac{dy}{dx} = \sin x + \cos x$$

#### (ii)**P**. **D**. **E**:

A differential equation which involves two or more independent variables & partial derivatives w.r.t them is known as P.D.E.

$$Eg: \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

#### Order of a D.E:

The order of a D.E is the order of the highest derivaties appearing in it.

$$Eg: \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$
The order is 2.

## **Degree of D.E**:

The degree of a D.E is the degree of the derivative of the highest order.

If the degree of a D.E is one then it is said to be a linear D.E.

# Higher order linear differential equation with constant co efficient:

The general form of a linear differential equation of  $n^{th}$  order is

$$\frac{d^{n}y}{dx^{n}} + a_{1}\frac{d^{(n-1)}y}{dx^{(n-1)}} + a_{2}\frac{d^{(n-2)}y}{dx^{(n-2)}} + \dots + a_{n-1}\frac{dy}{dx} + a_{n}y = b$$

$$\to (1)$$

where  $a_1, a_2, ..., a_n$  are constants & b is a function of x (or)constant.

The symbol D stands for differential operator.

$$ie, \frac{dy}{dx} = Dy, \frac{d^2y}{dx^2} = D^2y, ..., \frac{d^ny}{dx^n} = D^ny$$
  

$$\therefore (1) =>$$

$$(D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = b \rightarrow (2)$$

The complete solution of is given by

y = complementary function + particular integral

## **Complementary function:**

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Let m_1, m_2, m_3 ... be the roots of a n^{th} order
linear differential equation then,
(i) If the roots are real & distinct (ie m_1 \neq m_2 \neq \cdots)
            then |CF = Ae^{m_1x} + Be^{m_2x} + Ce^{m_3x} + \cdots|
(ii) If the roots are real & 2 are equal
(m_1 = m_2 = m(say) \& m_3 \neq m_4, ...)
             then CF = (Ax + B)e^{mx} + Ce^{m_3x} + \cdots
(iii) If 2 roots are imaginary (m = \alpha \pm i\beta) & others are real
      then |CF = e^{\alpha x}(A\cos\beta x + B\sin\beta x) + Ce^{m_3x} + \cdots|
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# Type I: RHS = 0

## Rules:

- 1. Find the roots from LHS of the given D. E
- 2. Write the CF.
- 3. Since the RHS = 0, PI = 0
- 4. Solution is y = CF.

#### **Problems:**

$$1. Solve y'' - 6y' + 25y = 0$$

Given, 
$$y'' - 6y' + 25y = 0$$
  
 $= > (D^2 - 6D + 25)y = 0$   
The auxillary equation is  $m^2 - 6m + 25 = 0$   
 $\therefore m = \frac{6 \pm \sqrt{36 - 100}}{2} = \frac{6 \pm \sqrt{-64}}{2}$   
 $= \frac{6 \pm i8}{2} = 3 \pm i4$   
(the roots are complex, here  $\alpha = 3 \& \beta = 4$ )

$$\therefore CF = e^{3x}(A\cos 4x + B\sin 4x)$$

$$\therefore$$
 Solution of d.e is  $y = CF(since\ RHS = 0)$ 

$$=> |y=e^{3x}(A\cos 4x+B\sin 4x)|$$

2. Solve  $(D^2 + 1)y = 0$ , gn y(0) = 0, y'(0) = 1.

Given, 
$$(D^2 + 1)y = 0$$
  
The auxillary equation is  $m^2 + 1 = 0$   
 $\Rightarrow m = \pm i$  (Here  $\alpha = 0 \& \beta = 1$ )  
 $\therefore CF = (A \cos x + B \sin x)$   
 $\therefore Solution of d.e is  $y = CF(since RHS = 0)$$ 

$$=> y(x) = (A\cos x + B\sin x)$$

$$Given y(0) = 0 \& y'(0) = 1$$

$$=> y(0) = A\cos 0 + B\sin 0 => A = 0$$

$$y'(x) = A(-\sin x) + B\cos x$$

$$=> y'(0) = -A\sin 0 + B\cos 0 => B = 1$$

$$\therefore y = 0(\cos x) + \sin x = > \boxed{y = \sin x}$$

3. Solve 
$$(D^3 + 3D^2 - 4)y = 0$$

4. Solve 
$$(D^4 - a^4)y = 0$$

The auxillary equation is 
$$m^4 - a^4 = 0$$
  
 $= > (m^2 + a^2)(m^2 - a^2) = 0$   
 $= > (m^2 + a^2) = 0 \& (m^2 - a^2) = 0$   
 $= > m = \pm ia \& m = \pm a$   
(Here two rt's are imaginary & 2 rt's are real & unequal)

$$\therefore CF = (A\cos ax + B\sin ax) + Ce^{ax} + De^{-ax}$$

$$\therefore$$
 Solution of d.e is  $y = CF(since RHS = 0)$ 

$$y = (A\cos ax + B\sin ax) + Ce^{ax} + De^{-ax}$$

# Type II: $RHS = e^{ax}$ Rules:

Let 
$$f(D)y = e^{ax}$$

1. Find CF.

$$2.PI = \frac{1}{f(D)}e^{ax}$$

3. Replace D by a

#### Case of failure:

If the denominator becomes zero,

- 4. Multiply numerator by x & differentiate the denominator ie,  $PI = \frac{x}{f'(D)}e^{ax}$
- 5. Repeat step 3.

#### **Problems:**

1. Solve 
$$(D-2)^2y = e^{2x}$$

Given, 
$$(D-2)^{2}y = e^{2x}$$
The auxillary equation is  $(m-2)^{2} = 0$ 

$$=> m = 2,2 \text{ (equal roots)}$$

$$\therefore \overline{CF = (Ax + B)e^{2x}}$$

$$PI = \frac{1}{(D-2)^{2}}e^{2x}$$

$$= \frac{1}{0}e^{2x} \text{ [Replace D by 2]}$$

$$\therefore PI = \frac{x}{2(D-2)}e^{2x} = \frac{x}{0}e^{2x} \text{ [Replace D by 2]}$$

$$= \frac{x^{2}}{2(1)}e^{2x}$$

$$PI = \frac{x^{2}}{2}e^{2x}$$

$$\therefore Solution of d.e is y = CF + PI$$
$$y = (Ax + B)e^{2x} + \frac{x^2}{2}e^{2x}$$

2. Solve 
$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2\cosh x$$

Given, 
$$(D^2 + 4D + 5)y = -2\left[\frac{e^x + e^{-x}}{2}\right]$$
  
 $(D^2 + 4D + 5)y = -(e^x + e^{-x})$   
The auxillary equation is  $m^2 + 4m + 5 = 0$   
 $=> m = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i$   
 $(Here \ \alpha = -2 \& \beta = 1)$   
 $\therefore CF = e^{-2x}(A\cos x + B\sin x)$ 

$$PI = \frac{1}{D^2 + 4D + 5} \left( -(e^x + e^{-x}) \right)$$

$$= -\frac{1}{D^2 + 4D + 5} e^x - \frac{1}{D^2 + 4D + 5} e^{-x}$$

$$= -\frac{1}{1 + 4 + 5} e^x - \frac{1}{1 - 4 + 5} e^{-x}$$

[Replace D by 1 in  $1^{st}$  term & -1 in  $2^{nd}$  term]

$$PI = -\frac{1}{10}e^x - \frac{1}{2}e^{-x}$$

 $\therefore$  Solution of d.e is y = CF + PI

$$y = e^{-2x}(A\cos x + B\sin x) - \frac{1}{10}e^x - \frac{1}{2}e^{-x}$$

3. Solve 
$$(D^2 + 2D + 2)y = 3 + 2e^x$$

Given, 
$$(D^2 + 2D + 2)y = 3e^0 + 2e^x$$
  
The a.e is  $m^2 + 2m + 2 = 0$   
 $=> m = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm i$   
 $(Here \ \alpha = -1 \ \& \ \beta = 1)$   
 $\therefore CF = e^{-x}(A\cos x + B\sin x)$   
 $PI = \frac{1}{D^2 + 2D + 2}[3e^0 + 2e^x]$   
 $= \frac{1}{D^2 + 2D + 2}3e^0 + \frac{1}{D^2 + 2D + 2}2e^x$   
 $= \frac{1}{0 + 0 + 2}3e^0 + \frac{1}{1 + 2 + 2}2e^x$   
[Replace D by 0 in 1st term & 1 in 2nd term]

$$\therefore$$
 Solution of d.e is  $y = CF + PI$ 

$$y = e^{-x}(A\cos x + B\sin x) + \frac{3}{2} + \frac{2}{5}e^{x}$$

# Type III: $RHS = \sin ax \ (or) \cos ax$ Rules:

Let 
$$f(D)y = \sin ax$$
 (or)  $\cos ax$ 

- 1. Find CF.
- $2.PI = \frac{1}{f(D)} [\sin ax] (or) \frac{1}{f(D)} [\cos ax]$
- 3. Replace  $D^2$  by  $-a^2$
- 4. Take conjugate of denominator & simplify

#### Case of failure:

After replacing, if the denominator becomes zero, 5. Multiply numerator by x & differentiate the denominator ie,

$$PI = \frac{x}{f'(D)} \sin ax \ (or) \frac{x}{f'(D)} \cos ax$$

6. Repeat step 3.

#### **Problems:**

1. Find the PI of  $(D^2 + 4)y = \cos 2x$ .

Given 
$$(D^2 + 4)y = \cos 2x$$
  

$$PI = \frac{1}{D^2 + 4}\cos 2x$$

$$= \frac{1}{-4 + 4}\cos 2x \left[Replace D^2 = -4\right]$$

$$= \frac{1}{0}\cos 2x \left(case \ of \ failure\right)$$

$$\therefore PI = \frac{x}{2D}\cos 2x$$

$$= \frac{x}{2} \frac{1}{D}\cos 2x = \frac{x}{2} \int \cos 2x \ dx$$

$$\therefore PI = \frac{x}{4}\sin 2x$$

2. Solve 
$$(D^2 - 4D + 3)y = \sin 3x \cos 2x$$

Given 
$$(D^2 - 4D + 3)y = \sin 3x \cos 2x$$
$$= \frac{1}{2} [\sin 5x + \sin x]$$
$$\left(\sin A \cos B = \frac{1}{2} [\sin(A + B) + \sin(A - B)]\right)$$

The a.e is  $m^2 - 4m + 3 = 0 => m = 3,1 (real \& unequal)$ 

$$\therefore |CF = Ae^{3x} + Be^{x}|$$

#### To find PI:

$$PI = \frac{1}{D^2 - 4D + 3} \left[ \frac{1}{2} \left[ \sin 5x + \sin x \right] \right]$$

$$= \frac{1}{2} \left[ \frac{1}{D^2 - 4D + 3} \sin 5x + \frac{1}{D^2 - 4D + 3} \sin x \right]$$

$$PI = \frac{1}{2} \left[ PI_1 + PI_2 \right]$$

$$PI_{1} = \frac{1}{D^{2} - 4D + 3} \sin 5x$$

$$= \frac{1}{-25 - 4D + 3} \sin 5x \quad [D^{2} \rightarrow -25]$$

$$= \frac{1}{-4D - 22} \sin 5x$$

$$= \frac{-4D + 22}{(-4D - 22)(-4D + 22)} \sin 5x$$

$$= \frac{-4D(\sin 5x) + 22 \sin 5x}{16D^{2} - 484}$$

$$= \frac{-20 \cos 5x + 22 \sin 5x}{16(-25) - 484} \quad [D^{2} \rightarrow -25]$$

$$= \frac{-20 \cos 5x + 22 \sin 5x}{-884}$$

$$PI_{1} = \frac{20 \cos 5x - 22 \sin 5x}{884}$$

$$PI_{2} = \frac{1}{D^{2} - 4D + 3} \sin x$$

$$= \frac{1}{-1 - 4D + 3} \sin x \quad [D^{2} \to -1]$$

$$= \frac{1}{-4D + 2} \sin x$$

$$= \frac{-4D + 2}{(-4D + 2)(-4D - 2)} \sin x$$

$$= \frac{-4D(\sin x) - 2\sin x}{16D^{2} - 4}$$

$$= \frac{-4\cos x - 2\sin x}{16(-1) - 4} \quad [D^{2} \to -1]$$

$$= \frac{-4\cos x - 2\sin x}{-20}$$

$$PI_{2} = \frac{2\cos x + \sin x}{10}$$

$$PI = \frac{1}{2} [PI_1 + PI_2]$$

$$PI = \frac{1}{2} \left[ \frac{20\cos 5x - 22\sin 5x}{884} + \frac{2\cos x + \sin x}{10} \right]$$

$$\therefore$$
 Solution of d.e is  $|y = CF + PI|$ 

3. 
$$Solve(D^2 + 6D + 8)y = cos^2x$$

Given 
$$(D^2 + 6D + 8)y = \cos^2 x$$
  
=  $\frac{1 + \cos 2x}{2} = \frac{e^0 + \cos 2x}{2}$   
 $\left(\cos^2 x = \frac{1 + \cos 2x}{2}\right)$ 

#### To find CF:

a.e. is 
$$m^2 + 6m + 8 = 0 => m = -4, -2$$
  

$$\therefore CF = Ae^{-4x} + Be^{-2x}$$

$$PI = \frac{1}{D^2 + 6D + 8} \left[ \frac{e^0 + \cos 2x}{2} \right]$$

$$= \frac{1}{2} \left[ \frac{1}{D^2 + 6D + 8} e^0 + \frac{1}{D^2 + 6D + 8} \cos 2x \right]$$

$$\therefore PI = \frac{1}{2} [PI_1 + PI_2]$$

$$PI_1 = \frac{1}{D^2 + 6D + 8} e^0 = \frac{1}{8} e^0$$

$$PI_2 = \frac{1}{D^2 + 6D + 8} \cos 2x$$

$$= \frac{1}{-4 + 6D + 8} \cos 2x \ [D^2 \to -4]$$

$$= \frac{1}{6D + 4} \cos 2x = \frac{6D - 4}{(6D + 4)(6D - 4)} \cos 2x$$

$$= \frac{6D(\cos 2x) - 4(\cos 2x)}{36D^2 - 16}$$

$$= \frac{6(-2\sin 2x) - 4(\cos 2x)}{36(-4) - 16}$$

$$= \frac{-12\sin 2x - 4\cos 2x}{-160}$$

$$= \frac{-160}{40}$$

$$PI_2 = \frac{3\sin 2x + \cos 2x}{40}$$

$$PI = \frac{1}{2}[PI_1 + PI_2]$$

$$PI = \frac{1}{2} \left[ \frac{1}{8} + \frac{3\sin 2x + \cos 2x}{40} \right]$$

$$\therefore Solution of d.e is | y = CF + PI |$$

# Type IV: $RHS = x^m$

# **Rules:**

Let 
$$f(D)y = x^m$$

1. Find CF.

$$2.PI = \frac{1}{f(D)}[x^m]$$

3. Rewrite denominator of the form

$$[1 \pm \varphi(D)] \quad ie \ PI = \frac{1}{1 \pm \varphi(D)} [x^m]$$

4. 
$$PI = [1 \pm \varphi(D)]^{-1} x^m$$

5. Expand & simplify.

#### Note:

$$(i)(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$
  
 $(ii)(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$ 

#### **Problems:**

1. Find PI of 
$$(D^2 + 4)y = x^2$$

### **Solution:**

Given, 
$$(D^2 + 4)y = x^2$$

### To find PI:

$$PI = \frac{1}{D^2 + 4} x^2$$

$$= \frac{1}{4 \left[ 1 + \frac{D^2}{4} \right]} x^2$$

$$= \frac{1}{4} \left[ 1 + \frac{D^2}{4} \right]^{-1} x^2$$

$$= \frac{1}{4} \left[ 1 - \frac{D^2}{4} + \frac{D^4}{16} - \dots \right] x^2$$

$$= \frac{1}{4} \left[ x^2 - \frac{2}{4} + 0 - \dots \right] = \frac{1}{4} \left[ x^2 - \frac{1}{2} \right]$$

$$PI = \frac{1}{4} \left[ x^2 - \frac{1}{2} \right]$$

2. Solve 
$$(D^2 + 5D + 4)y = x^2 + 7x + 9$$

Given 
$$(D^2 + 5D + 4)y = x^2 + 7x + 9$$
  
 $a.e is m^2 + 5m + 4 = 0 \Rightarrow m = -4, -1$   

$$\therefore CF = Ae^{-4x} + Be^{-x}$$

$$PI = \frac{1}{D^2 + 5D + 4} [x^2 + 7x + 9]$$

$$= \frac{1}{4 \left[1 + \frac{D^2 + 5D}{4}\right]^{-1}} [x^2 + 7x + 9]$$

$$= \frac{1}{4} \left[1 + \frac{D^2 + 5D}{4}\right]^{-1} [x^2 + 7x + 9]$$

$$= \frac{1}{4} \left[1 - \left(\frac{D^2 + 5D}{4}\right) + \left(\frac{D^2 + 5D}{4}\right)^2 - \cdots\right] [x^2 + 7x + 9]$$

$$= \frac{1}{4} \left[1 - \left(\frac{D^2 + 5D}{4}\right) + \left(\frac{D^2 + 5D}{4}\right)^2 - \cdots\right] [x^2 + 7x + 9]$$

$$= \frac{1}{4} \left[ 1 - \frac{D^2}{4} - \frac{5D}{4} + \frac{25D^2}{16} \right] [x^2 + 7x + 9]$$

$$\{D[x^2 + 7x + 9] = 2x + 7,$$

$$D^2[x^2 + 7x + 9] = 2\}$$

$$PI = \frac{1}{4} \left[ x^2 + 7x + 9 - \frac{2}{4} - \frac{5(2x + 7)}{4} + \frac{25(2)}{16} \right]$$

$$= \frac{1}{4} \left[ x^2 + 7x + 9 - \frac{1}{2} - \frac{5x}{2} - \frac{35}{4} + \frac{25}{8} \right]$$

$$PI = \frac{1}{4} \left[ x^2 + \frac{9x}{2} + \frac{23}{8} \right]$$

$$\therefore Solution of d. e is \left[ \mathbf{y} = \mathbf{CF} + \mathbf{PI} \right]$$

3. 
$$solve(D^3 + 2D^2 + D)y = e^{2x} + x^2 + x$$

$$a.e \ is \ m^{3} + 2m^{2} + m = 0$$

$$=> m(m^{2} + 2m + 1) = 0 => m = 0, -1, -1$$
Here the roots are real and one is distinct
$$CF = Ae^{0x} + (Bx + C)e^{-x}$$

$$PI = \frac{1}{D^{3} + 2D^{2} + D}e^{2x} + x^{2} + x$$

$$= \frac{1}{D^{3} + 2D^{2} + D}e^{2x} + \frac{1}{D^{3} + 2D^{2} + D}(x^{2} + x)$$

$$PI = \frac{1}{D^{3} + 2D^{2} + D}e^{2x}$$

$$PI_{1} = \frac{1}{D^{3} + 2D^{2} + D}e^{2x}$$

$$= \frac{1}{8 + 2(4) + 2}e^{2x} \ [D \to 2]$$

$$PI_{1} = \frac{e^{2x}}{18}$$

$$PI_{2} = \frac{1}{D^{3} + 2D^{2} + D}(x^{2} + x)$$

$$= \frac{1}{D(1 + (D^{2} + 2D))}(x^{2} + x)$$

$$= \frac{1}{D}[1 + (D^{2} + 2D)]^{-1}(x^{2} + x)$$

$$= \frac{1}{D}[1 - (D^{2} + 2D) + (D^{2} + 2D)^{2} - \cdots](x^{2} + x)$$

$$\{D(x^{2} + x) = 2x + 1, D^{2}(x^{2} + x) = 2\}$$

$$= \frac{1}{D}[1 - D^{2} - 2D + 4D^{2}](x^{2} + x)$$

$$= \frac{1}{D}[1 + 3D^{2} - 2D](x^{2} + x)$$

$$= \frac{1}{D}[(x^{2} + x) + 3(2) - 2(2x + 1)]$$

$$= \frac{1}{D}[x^2 - 3x + 4]$$

$$PI_2 = \frac{x^3}{3} - \frac{3x^2}{2} + 4x$$

$$\therefore PI = PI_1 + PI_2$$

$$PI = \frac{e^{2x}}{18} + \frac{x^3}{3} - \frac{3x^2}{2} + 4x$$

 $\therefore Solution of d.e is \mathbf{y} = \mathbf{CF} + \mathbf{PI}$ 

4. Solve 
$$(D^4 + D^3 + D^2)y = 5x^2 + \cos x$$

a.e is 
$$m^4 + m^3 + m^2 = 0$$
  
 $=> m^2(m^2 + m + 1) = 0$   
 $=> m^2 = 0$ ,  $m^2 + m + 1 = 0$   
 $=> m = 0,0$ ,  $\frac{-1 \pm i\sqrt{3}}{2}$ 

$$CF = (Ax + B) + e^{-\frac{1}{2}x} \left[ C \cos \frac{\sqrt{3}}{2}x + D \sin \frac{\sqrt{3}}{2}x \right]$$

$$PI = \frac{1}{D^4 + D^3 + D^2} [5x^2 + \cos x]$$

$$= \frac{1}{D^2(D^2 + D + 1)} 5x^2 + \frac{1}{D^2(D^2 + D + 1)} \cos x$$

$$PI = PI_1 + PI_2$$

$$PI_1 = \frac{1}{D^2(D^2 + D + 1)} 5x^2$$

$$= \frac{1}{D^2} [1 + (D^2 + D)]^{-1} 5x^2$$

$$= \frac{1}{D^2} [1 - (D^2 + D) + (D^2 + D)^2 + \cdots] 5x^2$$

$$= \frac{1}{D^2} [1 - D^2 - D + D^2] 5x^2$$

$$\{D(5x^2) = 10x, D^2(5x^2) = 10\}$$

$$= \frac{1}{D^{2}} [5x^{2} - 10 - 10x + 10]$$

$$= \frac{1}{D^{2}} [5x^{2} - 10x]$$

$$= \frac{1}{D} \left[ 5\frac{x^{3}}{3} - 10\frac{x^{2}}{2} \right]$$

$$PI_{1} = 5\frac{x^{4}}{12} - 5\frac{x^{3}}{3}$$

$$PI_{2} = \frac{1}{D^{2}(D^{2} + D + 1)} \cos x$$

$$= \frac{1}{(-1)((-1) + D + 1)} \cos x \quad [D^{2} \to -1]$$

$$= \frac{1}{-D} \cos x = -\sin x$$

$$PI_{2} = -\sin x$$

$$\therefore PI = PI_{1} + PI_{2}$$

$$\therefore Solution of d. e is y = CF + PI$$

5. Solve 
$$(D^2 - 3D + 2)y = 2\cos(2x + 3) + 2e^x$$

$$a.e \ is \ m^2 - 3m + 2 = 0 \Rightarrow m = 2,1$$

$$CF = Ae^{2x} + Be^{x}$$

$$PI = \frac{1}{D^2 - 3D + 2} [2\cos(2x + 3) + 2e^{x}]$$

$$= \frac{1}{D^2 - 3D + 2} 2\cos(2x + 3) + \frac{1}{D^2 - 3D + 2} 2e^{x}$$

$$PI_1 = \frac{1}{D^2 - 3D + 2} 2\cos(2x + 3)$$

$$= 2\frac{1}{(-4) - 3D + 2} \cos(2x + 3) \ [D^2 \to -4]$$

$$= 2\frac{1}{-3D - 2} \cos(2x + 3)$$

$$= 2\frac{1}{-3D + 2} \cos(2x + 3)$$

$$= 2\frac{1}{-3D - 2} \cos(2x + 3)$$

$$= 2\frac{-3(-2\sin(2x+3)) + 2\cos(2x+3)}{-40}$$

$$= \frac{-6\sin(2x+3) - 2\cos(2x+3)}{20}$$

$$PI_{1} = \frac{-3\sin(2x+3) - \cos(2x+3)}{10}$$

$$PI_{2} = \frac{1}{D^{2} - 3D + 2} 2e^{x}$$

$$= \frac{1}{1 - 3 + 2} 2e^{x} [D \to 1]$$

$$PI_{2} = \frac{x}{2D - 3} 2e^{x}$$

$$= \frac{x}{2 - 3} 2e^{x}$$

$$PI_{2} = -2xe^{x}$$

$$\therefore PI = PI_{1} + PI_{2}$$

$$\therefore Solution of d. e is [y = CF + PI]$$

# Type V: $RHS = e^{ax}V$ , where V is some function of x Rules:

Let 
$$f(D)y = e^{ax}V$$

- 1. Find CF
- $2.PI = \frac{1}{f(D)}e^{ax}V$ 
  - $3.PI = e^{ax} \frac{1}{f(D+a)} V$

4. Apply type III if V is  $\sin ax$  (or)  $\cos ax$  (or) type IV if V is  $x^m$ 

#### **Problems:**

1. Solve 
$$(D^2 - 2D + 5)y = e^{2x} \sin x$$

a. 
$$e ext{ is } m^2 - 2m + 5 = 0 => m = 1 \pm 2i$$

$$CF = e^x [A \cos 2x + B \sin 2x]$$

$$PI = \frac{1}{D^2 - 2D + 5} e^{2x} \sin x$$

$$= e^{2x} \frac{1}{(D+2)^2 - 2(D+2) + 5} \sin x$$

$$= e^{2x} \frac{1}{D^2 + 4D + 4 - 2D - 4 + 5} \sin x$$

$$= e^{2x} \frac{1}{D^2 + 2D + 5} \sin x$$

$$= e^{2x} \frac{1}{-1 + 2D + 5} \sin x \quad [D^2 \to -1]$$

$$= e^{2x} \frac{1}{2D + 4} \sin x$$

$$= e^{2x} \frac{1}{2D - 4} \sin x$$

$$= e^{2x} \frac{2 \cos x - 4 \sin x}{4D^2 - 16} \sin x$$

$$= e^{2x} \frac{2 \cos x - 4 \sin x}{10} \quad [D^2 \to -1]$$

$$PI = e^{2x} \frac{2 \sin x - \cos x}{10}$$

$$\therefore Solution of d. e is [y = CF + PI]$$

2. Solve 
$$(D^2 - 4D + 3)y = 2xe^{3x} + 3e^x \cos 2x$$

$$a.e \ is \ m^2 - 4m + 3 = 0 \Rightarrow m = 1,3$$

$$CF = Ae^x + Be^{3x}$$

$$PI = \frac{1}{D^2 - 4D + 3} 2xe^{3x} + \frac{1}{D^2 - 4D + 3} 3e^x \cos 2x$$

$$PI_1 = \frac{1}{D^2 - 4D + 3} 2xe^{3x}$$

$$= 2e^{3x} \frac{1}{(D+3)^2 - 4(D+3) + 3} x [D \to D+3]$$

$$= 2e^{3x} \frac{1}{D^2 + 6D + 9 - 4D - 12 + 3} x$$

$$= 2e^{3x} \frac{1}{D^2 + 2D} x = \frac{2e^{3x}}{2D} \left[1 + \frac{D}{2}\right]^{-1} x$$

$$= \frac{e^{3x}}{D} \left[1 - \frac{D}{2}\right] x \{D(x) = 1\}$$

$$= \frac{e^{3x}}{D} \left[ x - \frac{1}{2} \right]$$

$$PI_1 = e^{3x} \left[ \frac{x^2}{2} - \frac{x}{2} \right]$$

$$PI_2 = \frac{1}{D^2 - 4D + 3} 3e^x \cos 2x$$

$$= 3e^x \frac{1}{(D+1)^2 - 4(D+1) + 3} \cos 2x$$

$$[D \to D+1]$$

$$= 3e^x \frac{1}{D^2 - 2D} \cos 2x$$

$$= 3e^x \frac{1}{(-4) - 2D} \cos 2x [D^2 \to -4]$$

$$= -3e^x \frac{2D - 4}{(2D+4)(2D-4)} \cos 2x$$

$$= -3e^x \frac{2D - 4}{4D^2 - 16} \cos 2x$$

$$= -3e^{x} \frac{2(-2\sin 2x) - 4\cos 2x}{4(-4) - 16} [D^{2} \rightarrow -4]$$

$$= -3e^{x} \frac{4\sin 2x + 4\cos 2x}{32}$$

$$= -3e^{x} \frac{3e^{x}[\sin 2x + 4\cos 2x]}{32}$$

$$PI_2 = -\frac{3}{8}e^x[\sin 2x + \cos 2x]$$

$$PI = PI_1 + PI_2$$

$$\therefore$$
 Solution of d.e is  $y = CF + PI$ 

3. Solve 
$$(D^2 - 6)y = e^x \sin^2 x$$

a. 
$$e$$
 is  $m^2 - 6 = 0 => m = \pm \sqrt{6}$ 

$$CF = Ae^{\sqrt{6}x} + Be^{-\sqrt{6}x}$$

$$PI = \frac{1}{D^2 - 6}e^x \left[ \frac{1 - \cos 2x}{2} \right]$$

$$= \frac{1}{D^2 - 6} \frac{e^x}{2} - \frac{1}{D^2 - 6} \frac{e^x \cos 2x}{2}$$

$$PI_1 = \frac{1}{D^2 - 6} \frac{e^x}{2} = \frac{1}{2} \frac{1}{1 - 6} e^x [D \to 1]$$

$$PI_1 = -\frac{e^x}{10}$$

$$PI_2 = \frac{1}{D^2 - 6} \frac{e^x \cos 2x}{2}$$

$$= \frac{e^x}{2} \frac{1}{(D+1)^2 - 6} \cos 2x$$

$$= \frac{e^x}{2} \frac{1}{D^2 + 2D - 5} \cos 2x$$

$$= \frac{e^x}{2} \frac{1}{2D - 9} \cos 2x [D^2 \to -4]$$

$$= \frac{e^x}{2} \frac{2D + 9}{4D^2 - 81} \cos 2x$$

$$= \frac{e^x}{2} \frac{2(-2\sin 2x) + 9(\cos 2x)}{-97} \quad [D^2 \to -4]$$

$$PI_2 = \frac{e^x}{194} [4\sin 2x - 9\cos 2x]$$

$$\therefore PI = PI_1 - PI_2$$

$$\therefore Solution of d.e is  $\mathbf{v} = \mathbf{CF} + \mathbf{PI}$$$

 $\therefore$  Solution of d.e is y = CF + PI

# Type VI: RHS = $x^n \sin ax$ (or) $x^n \cos ax$ Rules:

- 1. Find CF
- $2.PI = \frac{1}{f(D)}x^n \sin ax (or) \frac{1}{f(D)}x^n \cos ax$
- 3.  $PI = \frac{1}{f(D)} IP(e^{iax}) x^n (or) \frac{1}{f(D)} RP(e^{iax}) x^n$
- 4. Apply type V and simplfy.

**Problems:** 

$$1. Solve y'' + 2y' + y = x \cos x$$

$$given (D^{2} + 2D + 1)y = x \cos x$$

$$a.e is m^{2} + 2m + 1 = 0 \Rightarrow m = -1, -1$$

$$CF = (Ax + B)e^{-x}$$

$$PI = \frac{1}{D^{2} + 2D + 1} x \cos x$$

$$= \frac{1}{D^{2} + 2D + 1} RPe^{ix}x$$

$$= RP \frac{1}{D^{2} + 2D + 1} e^{ix}x$$

$$= RPe^{ix} \frac{1}{(D+i)^{2} + 2(D+i) + 1} x$$

$$= RPe^{ix} \frac{1}{D^{2} + 2iD - 1 + 2D + 2i + 1} x$$

$$= RPe^{ix} \frac{1}{D^2 + 2D(i+1) + 2i} x$$

$$= RPe^{ix} \frac{1}{2i \left(1 + \frac{D^2 + 2D(i+1)}{2i}\right)} x$$

$$= RPe^{ix} \frac{1}{2i} \left(1 + \frac{D^2 + 2D(i+1)}{2i}\right)^{-1} x$$

$$= RPe^{ix} \frac{1}{2i} \left(1 - \frac{D^2 + 2D(i+1)}{2i} + \cdots\right) x$$

$$= RPe^{ix} \frac{1}{2i} \left(x - \frac{2(i+1)}{2i}\right) [D(x) = 1]$$

$$= RPe^{ix} \frac{1}{2i} \left(x - 1 - \frac{1}{i}\right)$$

$$= RP(\cos x + i \sin x) \frac{1}{-2} (2i(x-1) - 1)$$

$$= -\frac{1}{2} (-\cos x - (x-1)\sin x)$$

$$PI = \frac{1}{2}(\cos x + (x - 1)\sin x)$$

 $\therefore$  Solution of d.e is y = CF + PI

2. Solve 
$$(D^2 + 4)y = x^2 \sin 2x$$

a. 
$$e$$
 is  $m^2 + 4 = 0 \Rightarrow m = \pm 2i$   
 $CF = A \cos 2x + B \sin 2x$   

$$PI = \frac{1}{D^2 + 4} x^2 \sin 2x$$

$$= \frac{1}{D^2 + 4} x^2 (IPe^{i2x})$$

$$= IP \frac{1}{D^2 + 4} x^2 e^{i2x}$$

$$= IPe^{i2x} \frac{1}{(D + 2i)^2 + 4} x^2 [D \to D + 2i]$$

$$= IPe^{i2x} \frac{1}{D^2 + 4iD - 4 + 4} x^2$$

$$= IPe^{i2x} \frac{1}{D^2 + 4iD} x^2$$

$$= IP \frac{e^{i2x}}{4iD} \left[ 1 + \frac{D}{4i} \right]^{-1} x^2$$

$$= IP \frac{e^{i2x}}{4iD} \left[ 1 - \frac{D}{4i} + \left( \frac{D}{4i} \right)^2 + \cdots \right] x^2$$

$$= IP \frac{e^{i2x}}{4iD} \left[ x^2 - \frac{2x}{4i} + \frac{2}{-16} \right]$$

$$\{ D(x^2) = 2x, D^2(x^2) = 2 \}$$

$$= IP \frac{e^{i2x}}{4iD} \frac{[8ix^2 - 4x - i]}{8i}$$

$$= IP \frac{e^{i2x}}{-32D} \left[ 8ix^2 - 4x - i \right]$$

$$= -\frac{1}{32} IP(\cos 2x + i \sin 2x) \left[ \left( 8\frac{x^3}{3} - x \right)i - 2x^2 \right]$$

$$= -\frac{1}{32} \left[ \left( 8\frac{x^3}{3} - x \right) \cos 2x - 2x^2 \sin 2x \right]$$

$$PI = -\frac{1}{32} \left[ \left( 8\frac{x^3}{3} - x \right) \cos 2x - 2x^2 \sin 2x \right]$$

$$\therefore$$
 Solution of d.e is  $y = CF + PI$ 

3. Find PI of 
$$(D^2 - 2D + 1)y = xe^x \sin x$$

$$PI = \frac{1}{D^2 - 2D + 1} x e^x \sin x$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 1} x \sin x$$

$$= e^x \frac{1}{D^2 + 2D + 1 - 2D - 2 + 1} x \sin x$$

$$= e^x \frac{1}{D^2} x \sin x$$

$$= e^x \frac{1}{D} [-x \cos x + \sin x]$$

{using bernoulli formula,  
consider 
$$u = x \& dv = \sin x$$
}  
 $= e^x[-x \sin x - \cos x - \cos x]$   
 $PI = -e^x[x \sin x + 2 \cos x]$ 

## Cauchy's Linear Differential Equation:

An equation of the form

$$x^{n} \frac{d^{n} y}{dx^{n}} + a_{1} x^{n-1} \frac{d^{(n-1)} y}{dx^{(n-1)}} + \dots + a_{n-1} x \frac{dy}{dx} + a_{n} y = b \longrightarrow (1)$$

where  $a_1, a_2, ..., a_n$  are constants & b is a function of x (or) constant is called Cauchy's linear equation.

(1) can be reduced to linear d.e with constant co efficient by substituting

$$x = e^{z} = z = \log x$$

$$x \frac{dy}{dx} = D'y \text{ where } D' = \frac{d}{dz}$$

$$x^{2} \frac{d^{2}y}{dx^{2}} = D'(D' - 1)y$$

$$x^{3} \frac{d^{3}y}{dx^{3}} = D'(D' - 1)(D' - 2)y \text{ & so on.}$$

#### **Problems:**

1. Solve 
$$x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = 0$$

Given 
$$[x^{2}D^{2} - xD + 1]y = 0$$
  
Put  $x = e^{z} \Rightarrow z = \log x$   
 $xD = D', x^{2}D^{2} = D'(D' - 1)$ where  $D' = \frac{d}{dz}$   
 $\therefore [D'(D' - 1) - D' + 1]y = 0$   
 $\Rightarrow [D'^{2} - D' - D' + 1]y = 0$   
 $\Rightarrow [D'^{2} - 2D' + 1]y = 0$   
 $a.e \text{ is } m^{2} - 2m + 1 = 0$   
 $\Rightarrow m = 1,1$   
 $CF = (Az + B)e^{z}$   
 $\Rightarrow \text{soln is } y = CF \text{ (since RHS} = 0)$   
 $\therefore y = (Az + B)e^{z}$   
 $\Rightarrow y = (Az + B)e^{z}$ 

2. solve 
$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - 9y = 10 + \frac{5}{x^2}$$

Given 
$$[x^{2}D^{2} + xD - 9]y = 10 + \frac{5}{x^{2}}$$
  
 $Put \ x = e^{z} => z = \log x$   
 $xD = D', x^{2}D^{2} = D'(D' - 1)where \ D' = \frac{d}{dz}$   
 $[D'(D' - 1) + D' - 9]y = 10e^{0} + \frac{5}{(e^{z})^{2}}$   
 $[(D')^{2} - 9]y = 10e^{0} + 5e^{-2z}$   
 $a.e \ is \ m^{2} - 9 = 0 => m = \pm 3$   
 $CF = Ae^{3z} + Be^{-3z}$   
 $PI = PI_{1} + PI_{2}$   
 $PI_{1} = \frac{1}{(D')^{2} - 9} 10e^{0} = \frac{10}{-9} e^{0} \ [D' \to 0]$ 

$$PI_{1} = -\frac{10}{9}$$

$$PI_{2} = \frac{1}{(D')^{2} - 9} 5e^{-2z}$$

$$= 5\frac{1}{4 - 9}e^{-2z} [D' \to -2]$$

$$PI_{2} = -e^{-2z}$$

$$\therefore Soln \ is \ y = CF + PI$$

$$= > y = Ae^{3z} + Be^{-3z} - \frac{10}{9} - e^{-2z}$$

$$= > y = Ae^{3\log x} + Be^{-3\log x} - \frac{10}{9} - e^{-2\log x}$$

$$= > \boxed{y = Ax^{3} + Bx^{-3} - \frac{10}{9} - x^{-2}}$$

3. Solve 
$$[x^2D^2 - 3xD + 4]y = (\log x)^2 + \cos(\log x)$$

Solution:  
Given 
$$[x^2D^2 - 3xD + 4]y = (\log x)^2 + \cos(\log x)$$
  
 $Put \ x = e^z => z = \log x$   
 $xD = D', x^2D^2 = D'(D' - 1)where \ D' = \frac{d}{dz}$   
 $[D'(D' - 1) - 3D' + 4]y = (z)^2 + \cos(z)$   
 $=> [(D')^2 - 4D' + 4]y = z^2 + \cos z$   
 $a.e \ is \ m^2 - 4m + 4 = 0 => m = 2,2$   
 $CF = (Az + B)e^{2z}$   
 $PI = PI_1 + PI_2$   
 $PI_1 = \frac{1}{(D')^2 - 4D' + 4}z^2$  (type IV)  
 $= \frac{1}{4} \left[1 + \frac{(D')^2 - 4D'}{4}\right]^{-1}z^2$ 

$$= \frac{1}{4} \left[ 1 - \frac{(D')^2 - 4D'}{4} + \left( \frac{(D')^2 - 4D'}{4} \right)^2 \right] z^2$$

$$= \frac{1}{4} \left[ 1 - \frac{(D')^2}{4} + D' + D'^2 \right] z^2$$

$$\{ D'(z^2) = 2z, (D')^2 (z^2) = 2 \}$$

$$PI_1 = \frac{1}{4} \left[ z^2 + \frac{3}{2} + 2z \right]$$

$$PI_2 = \frac{1}{(D')^2 - 4D' + 4} \cos z \ (type \ III)$$

$$= \frac{1}{-1 - 4D' + 4} \cos z \ [(D')^2 \rightarrow -1]$$

$$= \frac{1}{-4D' + 3} \cos z = \frac{-4D' - 3}{16(D')^2 - 9} \cos z$$

$$PI_2 = \frac{4\sin z - 3\cos z}{-25}$$

 $\therefore$  Soln is y = CF + PI

$$=> y = (Az + B)e^{2z} + \frac{1}{4}\left[z^2 + \frac{3}{2} + 2z\right] - \left[\frac{4\sin z - 3\cos z}{25}\right]$$

$$y = (A \log x + B)x^{2} + \frac{1}{4} \left[ (\log x)^{2} + \frac{3}{2} + 2 \log x \right]$$

$$- \left[ \frac{4 \sin(\log x) - 3 \cos(\log x)}{25} \right]$$

4. Solve 
$$[x^2D^2 - xD + 1]y = \left[\frac{\log x}{x}\right]^2 + x \sin(\log x)$$

given 
$$[x^2D^2 - xD + 1]y = \left[\frac{\log x}{x}\right]^2 + x\sin(\log x)$$
  
Put  $x = e^z = > z = \log x$ 

$$xD = D', x^2D^2 = D'(D' - 1)$$
 where  $D' = \frac{d}{dz}$ 

$$[D'(D'-1) - D' + 1]y = \left(\frac{z}{e^{z}}\right)^{2} + e^{z} \sin z$$

$$= > [(D')^{2} - 2D' + 1]y = e^{-2z}z^{2} + e^{z} \sin z$$

$$a. e \text{ is } m^{2} - 2m + 1 = 0 \Rightarrow m = 1, 1$$

$$CF = (Az + B)e^{z}$$

$$PI = PI_{1} + PI_{2}$$

$$PI_{1} = \frac{1}{(D')^{2} - 2D' + 1}e^{-2z}z^{2} \text{ (type V)}$$

$$= e^{-2z}\frac{1}{(D' - 2)^{2} - 2(D' - 2) + 1}z^{2}$$

$$[D' \to D' - 2]$$

$$= e^{-2z}\frac{1}{(D')^{2} - 6D' + 9}z^{2} \text{ (type IV)}$$

$$= \frac{e^{-2z}}{9}\left[1 + \frac{(D')^{2} - 6D'}{9}\right]^{-1}z^{2}$$

$$= \frac{e^{-2z}}{9} \left[ 1 - \frac{(D')^2 - 6D'}{9} + \frac{36(D')^2}{81} \right] z^2$$

$$= \frac{e^{-2z}}{9} \left[ z^2 - \frac{2}{9} + \frac{2(2z)}{3} + \frac{36(2)}{81} \right]$$

$$= \frac{e^{-2z}}{9} \left[ z^2 - \frac{2}{9} + \frac{4z}{3} + \frac{8}{9} \right]$$

$$PI_1 = \frac{e^{-2z}}{9} \left[ z^2 + \frac{4z}{3} + \frac{6}{9} \right]$$

$$PI_2 = \frac{1}{(D')^2 - 2D' + 1} e^z \sin z \ (type \ V)$$

$$= e^z \frac{1}{(D' + 1)^2 - 2(D' + 1) + 1} \sin z$$

$$[D' \to D' + 1]$$

$$= e^z \frac{1}{(D')^2} \sin z \ (type \ III)$$

$$= e^z \frac{1}{(-1)} \sin z \ [(D')^2 \to -1]$$

$$PI_2 = -e^z \sin z$$

$$\therefore Soln is y = CF + PI$$

$$y = (Az + B)e^z + \frac{e^{-2z}}{9} \left[ z^2 + \frac{4z}{3} + \frac{6}{9} \right] - e^z \sin z$$

$$y = (A \log x + B)x + \frac{x^{-2}}{9} \left[ (\log x)^2 + \frac{4 \log x}{3} + \frac{6}{9} \right]$$

$$-x \sin(\log x)$$

5. Solve  $[x^2D^2 + xD + 1]y = \log x \sin(\log x)$ 

$$Gn [x^2D^2 + xD + 1]y = \log x \sin(\log x)$$
  
 $Put x = e^z => z = \log x$   
 $xD = D', x^2D^2 = D'(D' - 1)where D' = \frac{d}{dz}$   
 $[D'(D' - 1) + D' + 1]y = z \sin z$   
 $=> [(D')^2 + 1]y = z \sin z$   
 $a. e m^2 + 1 = 0 => m = \pm i$   
 $CF = A \cos z + B \sin z$ 

$$PI = \frac{1}{(D')^{2} + 1} z \sin z \ (type \ VI)$$

$$= \frac{1}{(D')^{2} + 1} z (IP \ e^{iz})$$

$$= IPe^{iz} \frac{1}{(D' + i)^{2} + 1} z [D' \to D' + i]$$

$$= IPe^{iz} \frac{1}{(D')^{2} + 2iD' - 1 + 1} z$$

$$= IPe^{iz} \frac{1}{(D')^{2} + 2iD'} z$$

$$= IPe^{iz} \frac{1}{2iD'} \left[ 1 + \frac{D'}{2i} \right]^{-1} z$$

$$= IPe^{iz} \frac{1}{2iD'} \left[ 1 - \frac{D'}{2i} \right] z$$

$$= IPe^{iz} \frac{1}{2iD'} \left[ z - \frac{1}{2i} \right]$$

$$= IPe^{iz} \frac{1}{2i} \left[ \frac{z^2}{2} - \frac{z}{2i} \right]$$

$$= IP(\cos z + i \sin z) \frac{1}{-4} [iz^2 - z]$$

$$PI = -\frac{1}{4} [z^2 \cos z - z \sin z]$$

$$\therefore Soln \ is \ y = CF + PI$$

$$y = A \cos z + B \sin z - \frac{1}{4} [z^2 \cos z - z \sin z]$$

$$y = A\cos(\log x) + B\sin(\log x) - \frac{1}{4}[(\log x)^2\cos(\log x) - \log x\sin(\log x)]$$

# Legendre's Differential Equation:

An equation of the form

$$(ax + b)^{n} \frac{d^{n}y}{dx^{n}} + k_{1}(ax + b)^{n-1} \frac{d^{(n-1)}y}{dx^{(n-1)}} + \dots + k_{n-1}(ax + b) \frac{dy}{dx}$$
$$+ k_{n}y = b \rightarrow (1)$$

where  $k_1, k_2, ..., k_n$  are constants & b is a function of x (or)constant is called Legendre's linear equation.

(1) can be reduced to linear d.e with constant co efficient by substituting

$$ax + b = e^{z} = z = \log(ax + b)$$

$$ax + b\frac{dy}{dx} = aD'y \text{ where } D' = \frac{d}{dz}$$

$$(ax + b)^{2}\frac{d^{2}y}{dx^{2}} = a^{2}D'(D' - 1)y$$

$$(ax + b)^{3}\frac{d^{3}y}{dx^{3}} = a^{3}D'(D' - 1)(D' - 2)y$$

& so on.

#### **Problems:**

1. Solve 
$$[(x+2)^2D^2 - (x+2)D + 1]y = 3x + 4$$

$$gn, [(x+2)^{2}D^{2} - (x+2)D + 1]y = 3x + 4$$

$$Sub \ x + 2 = e^{z} => x = e^{z} - 2$$

$$z = \log(x+2)$$

$$(x+2)D = D', (x+2)^{2}D^{2} = D'(D'-1)where D' = \frac{d}{dz}$$

$$=> [D'(D'-1) - D' + 1]y = 3(e^{z} - 2) + 4$$

$$=> [(D')^{2} - 2D' + 1]y = 3e^{z} - 2e^{0}$$

$$a. e is m^{2} - 2m + 1 = 0 => m = 1,1$$

$$CF = (Az + B)e^{z}$$

$$PI = PI_{1} + PI_{2}$$

$$PI_{1} = \frac{1}{(D')^{2} - 2D' + 1} 3e^{z} (type II)$$

$$= \frac{1}{1 - 2 + 1} 3e^{z} [D' \to 1]$$

$$\therefore PI_{1} = \frac{1}{2D' - 2} 3e^{z} = \frac{z}{2 - 2} 3e^{z} [D' \to 1]$$

$$\therefore PI_{1} = \frac{z^{2}}{2} 3e^{z}$$

$$PI_{2} = \frac{1}{(D')^{2} - 2D' + 1} (-2)e^{0} (type II)$$

$$= \frac{1}{0 - 0 + 1} (-2)e^{0}$$

$$PI_{2} = -2$$

$$\therefore Soln is y = CF + PI$$

$$y = (Az + B)e^{z} + \frac{z^{2}}{2} 3e^{z} - 2$$

$$y = (A \log(x + 2) + B)(x + 2) + \frac{3}{2} (\log(x + 2))^{2} (x + 2) - 2$$

2. Solve  $[(3x + 2)^2D^2 + 3(3x + 2)D - 36]y = 3x^2 + 4x + 1$  Solution:

$$gn [(3x+2)^{2}D^{2} + 3(3x+2)D - 36]y = 3x^{2} + 4x + 1$$

$$Put 3x + 2 = e^{z} = > x = \frac{e^{z} - 2}{3}$$

$$z = \log(3x+2)$$

$$(3x + 2)D = 3D', (3x + 2)^2D^2 = 9D'(D' - 1)$$
 where  $D' = \frac{a}{dz}$   
=>  $[9D'(D' - 1) + 9D' - 36]y = 3\frac{(e^z - 2)^2}{9} + 4\frac{e^z - 2}{3} + 1$ 

$$D'(D'-1) + 9D' - 36]y = 3\frac{9}{9} + 4\frac{3}{3} + 1$$
$$[9(D')^2 - 9D' + 9D' - 36]y$$

$$= \frac{1}{3}[e^{2z} - 4e^z + 4] + \frac{4}{3}(e^z - 2) + 1$$

$$[9(D')^2 - 36]y = \frac{e^{2z}}{3} - \frac{1}{3}$$

a. 
$$e$$
 is  $9m^2 - 36 = 0 => m = \pm 2$ 

$$CF = Ae^{2z} + Be^{-2z}$$

$$PI = PI_1 + PI_2$$

$$PI_{1} = \frac{1}{9(D')^{2} - 36} \frac{e^{2z}}{3} (type \ II)$$

$$= \frac{1}{3} \frac{1}{36 - 36} e^{2z} [D' \to 2]$$

$$\therefore PI_{1} = \frac{1}{3} \frac{z}{18D'} e^{2z} = \frac{1}{3} \frac{z}{36} e^{2z} [D' \to 2]$$

$$\therefore PI_{1} = \frac{1}{108} z e^{2z}$$

$$\therefore PI_{2} = \frac{1}{9(D')^{2} - 36} \left( -\frac{e^{0}}{3} \right) (type \ II)$$

$$= -\frac{1}{3} \frac{1}{0 - 36} e^{0} = \frac{1}{108}$$

$$\therefore PI_{2} = \frac{1}{108}$$

$$\therefore Soln \ is \ y = CF + PI$$

$$y = Ae^{2z} + Be^{-2z} + \frac{1}{108}ze^{2z} + \frac{1}{108}$$
$$y = A(3x+2)^2 + B(3x+2)^{-2} + \frac{1}{108}\log(3x+2)(3x+2)^2 + \frac{1}{108}$$

3. Solve  $[(x+1)^2D^2 + (x+1)D + 1]y = 4\cos(\log(x+1))$ Solution:

$$Gn, [(x+1)^{2}D^{2} + (x+1)D + 1]y = 4\cos(\log(x+1))$$

$$Sub \ x + 1 = e^{z} => x = e^{z} - 1$$

$$z = \log(x+1)$$

$$(x+1)D = D', (x+1)^{2}D^{2} = D'(D'-1)where D' = \frac{d}{dz}$$

$$=> [D'(D'-1) + D' + 1]y = 4\cos z$$

$$[(D')^{2} + 1]y = 4\cos z$$

$$a.e \ is \ m^{2} + 1 = 0 => m = \pm i$$

$$CF = A\cos z + B\sin z$$

$$PI = \frac{1}{(D')^2 + 1} 4 \cos z \ (type \ III)$$

$$= \frac{1}{-1 + 1} 4 \cos z \ [(D')^2 \to -1]$$

$$\therefore PI = \frac{z}{2D'} 4 \cos z$$

$$= 2z \int \cos z \, dz$$

$$PI = 2z \sin z$$

$$\therefore Soln \ is \ y = CF + PI$$

$$y = A \cos z + B \sin z + 2z \sin z$$

$$y = A \cos(\log(x + 1)) + B \sin(\log(x + 1)) + 2(\log(x + 1)) \sin(\log(x + 1))$$

# Simultaneous first order linear equations with constant co efficient:

If x & y are 2 dependent variables & t is independent variable, then the pair of equation of the form

$$f_1(D)x + f_2(D)y = T_1 \rightarrow (1)$$
  
 $g_1(D)x + g_2(D)y = T_2 \rightarrow (2)$ 

where  $f_1$ ,  $f_2$ ,  $g_1$ ,  $g_2$  are polynomials in the operator

$$D = \frac{d}{dt} \& T_1 \& T_2 \text{ are functions of t is called}$$

simultaneous d.e.

To solve (1) and (2), we proceed as in solving simultaneous algebric equations.

Problems:  
1. Solve the simultaneous eqn's
$$\frac{dx}{dt} + 2x + 3y = 2e^{2t}$$

$$\frac{dy}{dt} + 3x + 2y = 0$$
Solution:  
Gn,
$$\frac{dx}{dt} + 2x + 3y = 2e^{2t} => Dx + 2x + 3y = 2e^{2t}$$

$$=> (D+2)x + 3y = 2e^{2t} \rightarrow (1)$$

$$\frac{dy}{dt} + 3x + 2y = 0 => Dy + 3x + 2y = 0$$

$$=> 3x + (D+2)y = 0 \rightarrow (2)$$
solve (1) & (2)

$$(1) \times (D+2) => (D+2)^2 x + 3(D+2)y = 2(D+2)e^{2t} \to (3)$$

$$(2) \times 3 => 9x + 3(D+2)y = 0 \rightarrow (4)$$

$$(3) - (4) => (D+2)^{2}x - 9x = 2(D+2)e^{2t}$$

$$=> [D^{2} + 4D + 4 - 9]x = 2[2e^{2t} + 2e^{2t}]$$

$$= > [D^{2} + 4D - 5]x = 8e^{2t}$$

$$a.e \ is \ m^{2} + 4m - 5 = 0 = > m = -5,1$$

$$CF = Ae^{-5t} + Be^{t}$$

$$PI = \frac{1}{D^{2} + 4D - 5} 8e^{2t} \ (type \ II)$$

$$= \frac{1}{4 + 8 - 5} 8e^{2t} \ [D \rightarrow 2]$$

$$PI = \frac{8}{7}e^{2t}$$

$$= CF + PI$$

$$\therefore Soln \ is \ x = CF + PI$$

$$=> \left| x = Ae^{-5t} + Be^t + \frac{8}{7}e^{2t} \right| \to (5)$$

Diff (5) w.r.t.t  $\frac{dx}{dt} = -5Ae^{-5t} + Be^{t} + \frac{16}{7}e^{2t} \to (6)$ Sub (5) & (6) in (1) we get,

$$(1) = \frac{dx}{dt} + 2x + 3y = 2e^{2t}$$

$$-5Ae^{-5t} + Be^{t} + \frac{16}{7}e^{2t} + 2\left[Ae^{-5t} + Be^{t} + \frac{8}{7}e^{2t}\right] + 3y = 2e^{2t}$$

$$-5Ae^{-5t} + Be^{t} + \frac{16}{7}e^{2t} + 2Ae^{-5t} + 2Be^{t} + \frac{16}{7}e^{2t} + 3y = 2e^{2t}$$

$$-3Ae^{-5t} + 3Be^{t} + \frac{32}{7}e^{2t} + 3y = 2e^{2t}$$

$$3y = 2e^{2t} + 3Ae^{-5t} - 3Be^{t} - \frac{32}{7}e^{2t}$$

$$y = Ae^{-5t} - Be^{t} - \frac{6}{7}e^{2t}$$

Hence the solution are

$$x = Ae^{-5t} + Be^t + \frac{8}{7}e^{2t}$$
,  $y = Ae^{-5t} - Be^t - \frac{6}{7}e^{2t}$ 

2. Solve 
$$\frac{dx}{dt} + 2x - 3y = 5t$$
,  $\frac{dy}{dt} - 3x + 2y = 0$  given that  $x(0) = 0 \& y(0) = -1$ .

Gn, 
$$\frac{dx}{dt} + 2x - 3y = 5t \implies (D+2)x - 3y = 5t$$

$$\to (1)$$

$$\frac{dy}{dt} - 3x + 2y = 0 \implies -3x + (D+2)y = 0$$

$$\to (2)$$

$$(1) \times (D+2) \implies$$

$$(D+2)^2x - 3(D+2)y = 5(D+2)t \to (3)$$

$$(2) \times -3 \implies 9x - 3(D+2)y = 0 \to (4)$$

$$(3) - (4) \implies (D+2)^2x - 9x = 5(D+2)t$$

$$\implies [D^2 + 4D - 5]x = 5(1+2t)$$
a.  $e$  is  $m^2 + 4m - 5 = 0 \implies m = -5,1$ 

$$\boxed{CF = Ae^{-5t} + Be^t}$$

$$PI = \frac{1}{D^2 + 4D - 5} 5(1 + 2t)$$

$$= -\frac{5}{5} \frac{1}{\left[1 - \frac{D^2 + 4D}{5}\right]} (1 + 2t)$$

$$= -\left[1 - \frac{D^2 + 4D}{5}\right]^{-1} (1 + 2t)$$

$$= -\left[1 + \frac{D^2 + 4D}{5} + \cdots\right] (1 + 2t)$$

$$= -\left[(1 + 2t) + \frac{4}{5}(2)\right] \{D(1 + 2t) = 2\}$$

$$PI = -\left[2t + \frac{13}{5}\right]$$

$$\therefore Soln \ is \ x = CF + PI$$

$$= > x = Ae^{-5t} + Be^t - \left[2t + \frac{13}{5}\right] \rightarrow (5)$$

$$\frac{dx}{dt} = -5Ae^{-5t} + Be^{t} - 2 \to (6)$$

$$Sub (5) \& (6) in (1) we get,$$

$$(1) = > \frac{dx}{dt} + 2x - 3y = 5t$$

$$-5Ae^{-5t} + Be^{t} - 2 + 2\left[Ae^{-5t} + Be^{t} - \left[2t + \frac{13}{5}\right]\right] - 3y = 5t$$

$$-5Ae^{-5t} + Be^{t} - 2 + 2Ae^{-5t} + 2Be^{t} - 2\left[2t + \frac{13}{5}\right] - 3y = 5t$$

$$-3y = 5t + 3Ae^{-5t} - 3Be^{t} + 4t + \frac{36}{5}$$

$$y = -Ae^{-5t} + Be^{t} - 3t - \frac{12}{5} \to (7)$$

$$given that x(0) = 0 \& y(0) = -1$$

$$\therefore x(0) = Ae^{-5(0)} + Be^{0} - \left[0 + \frac{13}{5}\right] = 0$$

$$= > A + B = \frac{13}{5} \to (8)$$

$$&y(0) = -Ae^{0} + Be^{0} + 0 - \frac{12}{5} = -1$$
$$= > -A + B = \frac{7}{5} \to (9)$$

(8) + (9) => 
$$2B = \frac{13}{5} + \frac{7}{5} => 2B = 4$$
  
=>  $B = 2$   
Sub  $B = 2$  in (8) we get  
 $A = \frac{13}{5} - 2 => A = \frac{3}{5}$ 

∴ Solutions are

$$x = \frac{3}{5}e^{-5t} + 2e^{t} - \left[2t + \frac{13}{5}\right],$$
$$y = -\frac{3}{5}e^{-5t} + 2e^{t} - 3t - \frac{12}{5}$$

3. Solve  $Dx + y = \sin t$ ,  $x + Dy = \cos t$ gn that x = 2 & y = 0 at t = 0

Solution:

Given, 
$$Dx + y = \sin t \rightarrow (1)$$
  
 $\& x + Dy = \cos t \rightarrow (2)$   
 $(1) \times D => D^2x + Dy = D(\sin t) \rightarrow (3)$   
 $(2) \times 1 => x + Dy = \cos t \rightarrow (4)$   
 $(3) - (4) => D^2x - x = \cos t - \cos t$   
 $=> (D^2 - 1)x = 0$   
 $a.e \ is \ m^2 - 1 = 0 => m = \pm 1$   
 $CF = Ae^t + Be^{-t}$   
 $since \ RHS = 0 => PI = 0$   
 $\therefore soln \ is \ x = CF => \boxed{x = Ae^t + Be^{-t}} \rightarrow (5)$ 

Diff(5) w.r.t.t

$$\frac{dx}{dt} = Ae^t - Be^{-t} \to (6)$$

Sub (6) in (1)

$$Ae^t - Be^{-t} + y = \sin t$$

4. Solve 
$$Dx + 2x - 3y = t$$
,  $Dy - 3x + 2y = e^{2t}$ 

$$Gn, (D+2)x - 3y = t \to (1)$$

$$-3x + (D+2)y = e^{2t} \to (2)$$

$$(1) \times (D+2) => (D+2)^{2}x - 3(D+2)y = (D+2)t \to (3)$$

$$(2) \times 3 => -9x + 3(D+2)y = 3e^{2t} \to (4)$$

$$(3) + (4) => (D+2)^{2}x - 9x = (D+2)t + 3e^{2t}$$

$$[D^{2} + 4D - 5]x = 1 + 2t + 3e^{2t}$$

a. 
$$e$$
 is  $m^2 + 4m - 5 = 0 => m = -5,1$ 

$$CF = Ae^{-5t} + Be^{t}$$

$$PI_1 = \frac{1}{D^2 + 4D - 5}(1 + 2t)$$

$$= -\frac{1}{5} \left[1 - \frac{D^2 + 4D}{5}\right]^{-1}(1 + 2t)$$

$$= -\frac{1}{5} \left[1 - \frac{D^2 + 4D}{5}\right]^{-1}(1 + 2t)$$

$$= -\frac{1}{5} \left[1 + \frac{D^2 + 4D}{5} + \cdots\right](1 + 2t)$$

$$= -\frac{1}{5} \left[(1 + 2t) + \frac{4}{5}(2)\right] \{D(1 + 2t) = 2\}$$

$$PI = -\frac{1}{5} \left[2t + \frac{13}{5}\right]$$

$$PI_{2} = \frac{1}{D^{2} + 4D - 5} 3e^{2t}$$

$$= 3 \frac{1}{4 + 8 - 5} e^{2t}$$

$$PI_{2} = \frac{3}{7} e^{2t}$$

$$\therefore soln \ is \ x = CF + PI$$

$$x = Ae^{-5t} + Be^{t} - \frac{1}{5} \left[ 2t + \frac{13}{5} \right] + \frac{3}{7}e^{2t} \rightarrow (5)$$

$$Diff (5)w.r.t t$$

$$\frac{dx}{dt} = -5Ae^{-5t} + Be^{t} - \frac{2}{5} + \frac{6}{7}e^{2t} \to (6)$$
Sub (5) & (6) in (1) we get.

Sub (5) & (6) in (1) we get,  

$$(1) = > \frac{dx}{dt} + 2x - 3y = t$$

$$-5Ae^{-5t} + Be^{t} - \frac{2}{5} + \frac{6}{7}e^{2t} + 2Ae^{-5t} + 2Be^{t} - \frac{2}{5} \left[ 2t + \frac{13}{5} \right]_{77}$$

$$+ \frac{6}{7}e^{2t} - 3y = t$$

$$-3Ae^{-5t} + 3Be^{t} - \frac{4}{5}t + \frac{12}{7}e^{2t} - \frac{36}{25} - 3y = t$$

$$-3y = t + 3Ae^{-5t} - 3Be^{t} + \frac{4}{5}t - \frac{12}{7}e^{2t} + \frac{36}{25}$$

$$-3y = 3Ae^{-5t} - 3Be^{t} + \frac{9}{5}t - \frac{12}{7}e^{2t} + \frac{36}{25}$$

$$y = -Ae^{-5t} + Be^{t} - \frac{3}{5}t + \frac{4}{7}e^{2t} - \frac{12}{25}$$

Hence the solution is

$$x = Ae^{-5t} + Be^{t} - \frac{1}{5} \left[ 2t + \frac{13}{5} \right] + \frac{3}{7}e^{2t},$$

$$y = -Ae^{-5t} + Be^{t} - \frac{3}{5}t + \frac{4}{7}e^{2t} - \frac{12}{25}$$

# Method of variation of parameter:

Let us consider the second order d.e

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = X \to (1)$$

The CF of (1) is  $CF = Af_1 + Bf_2$ 

Let  $W = f_1 f_2' - f_2 f_1'$  be the Wronskian of the function.

Then 
$$PI = Pf_1 + Qf_2$$

where 
$$P = -\int \frac{f_2X}{W} dx \& Q = \int \frac{f_1X}{W} dx$$

$$\therefore y = CF + PI$$

## Note:

- $1. \int \tan ax \, dx = -\frac{1}{a} \log \cos ax$
- $2. \int \cot ax \, dx = \frac{1}{a} \log \sin ax$
- $3. \int \sec ax \, dx = \frac{1}{a} \log(\sec ax + \tan ax)$
- $4. \int \csc ax \ dx = -\frac{1}{a} \log(\csc ax + \cot ax)$

### **Problems:**

1. Solve  $(D^2 + 4)y = \cot 2x$  using method of variation of parameter.

$$Gn, (D^{2} + 4)y = \cot 2x$$

$$The \ a. e \ is \ m^{2} + 4 = 0 \Rightarrow m = \pm 2i$$

$$\therefore \boxed{CF = A \cos 2x + B \sin 2x} \quad (Af_{1} + Bf_{2})$$

$$Here \ f_{1} = \cos 2x \Rightarrow f_{1}' = -2 \sin 2x$$

$$f_{2} = \sin 2x \Rightarrow f_{2}' = 2 \cos 2x$$

$$W = f_{1}f_{2}' - f_{2}f_{1}' = \cos 2x (2 \cos 2x) - \sin 2x (-2 \sin 2x)$$

$$= 2\cos^{2}2x + 2\sin^{2}2x = 2$$

$$Here \ X = \cot 2x$$

$$Then \ PI = Pf_{1} + Qf_{2}$$

$$where \ P = -\int \frac{f_{2}X}{W} dx = -\int \frac{\sin 2x \cot 2x}{2} dx$$

$$= -\frac{1}{2} \int \sin 2x \frac{\cos 2x}{\sin 2x} dx$$

$$= -\frac{1}{2} \int \cos 2x \ dx$$

$$P = -\frac{\sin 2x}{4}$$

$$Q = \int \frac{f_1 X}{W} dx = \int \frac{\cos 2x \cot 2x}{2} dx$$

$$= \frac{1}{2} \int \frac{\cos^2 2x}{\sin 2x} dx$$

$$= \frac{1}{2} \int \frac{1 - \sin^2 2x}{\sin 2x} dx$$

$$= \frac{1}{2} \left| \int \csc 2x \, dx - \int \sin 2x \, dx \right|$$

$$= \frac{1}{2} \left[ -\frac{1}{2} \log(\csc 2x + \cot 2x) + \frac{1}{2} \cos 2x \right]$$

$$Q = -\frac{1}{4} \log(\csc 2x + \cot 2x) + \frac{1}{4} \cos 2x$$

$$PI = Pf_1 + Qf_2$$

$$= \left(-\frac{\sin 2x}{4}\right)\cos 2x$$

$$+ \left(-\frac{1}{4}\log(\csc 2x + \cot 2x) + \frac{1}{4}\cos 2x\right)\sin 2x$$

$$PI = -\frac{\sin 2x}{4}\log(\csc 2x + \cot 2x)$$

$$\therefore$$
 soln is  $y = CF + PI$ 

$$y = A\cos 2x + B\sin 2x - \frac{\sin 2x}{4} \log(\csc 2x + \cot 2x)$$

$$2. Solve (D^2 + a^2)y = \sec ax$$

$$Gn, (D^{2} + a^{2})y = \sec ax$$

$$The \ a. \ e \ is \ m^{2} + a^{2} = 0 \Rightarrow m = \pm ai$$

$$\therefore \boxed{CF = A \cos ax + B \sin ax} \quad (Af_{1} + Bf_{2})$$

$$Here \ f_{1} = \cos ax \Rightarrow f'_{1} = -a \sin ax$$

$$f_{2} = \sin ax \Rightarrow f'_{2} = a \cos ax$$

$$W = f_{1}f'_{2} - f_{2}f'_{1} = a \cos^{2}ax + a \sin^{2}ax = a$$

$$Here \ X = \sec ax$$

$$Then \ PI = Pf_{1} + Qf_{2}$$

$$where \ P = -\int \frac{f_{2}X}{W} dx = -\int \frac{\sin ax \sec ax}{a} dx$$

$$= -\frac{1}{a} \int \tan ax \ dx$$

$$= -\frac{1}{a} \left[ -\frac{1}{a} \log(\cos ax) \right]$$

$$P = \frac{1}{a^2} \log(\cos ax)$$

$$Q = \int \frac{f_1 X}{W} dx = \int \frac{\cos ax \sec ax}{a} dx$$

$$= \frac{1}{a} \int dx = \frac{x}{a}$$

$$Q = \frac{x}{a}$$

$$\therefore PI = Pf_1 + Qf_2 = \left(\frac{1}{a^2}\log(\cos ax)\right)\cos ax + \frac{x}{a}\sin ax$$

$$\therefore$$
 soln is  $y = CF + PI$ 

$$y = A \cos ax + B \sin ax + \left(\frac{1}{a^2} \log(\cos ax)\right) \cos ax + \frac{x}{a} \sin ax$$

3. 
$$solve(D^2 - 2D + 1)y = e^x \log x$$

Gn, 
$$(D^2 - 2D + 1)y = e^x \log x$$
  
 $Here \ m = 1,1$   
 $\therefore CF = (Ax + B)e^x = Axe^x + Be^x$   
 $Here \ f_1 = xe^x => f_1' = x(e^x) + e^x(1)$   
 $f_2 = e^x => f_2' = e^x$   
 $W = f_1f_2' - f_2f_1' = (xe^x)e^x - e^x(xe^x + e^x)$   
 $=> W = -e^{2x}$ 

Here 
$$X = e^x \log x$$

where 
$$P = -\int \frac{f_2 X}{W} dx = -\int \frac{e^x e^x \log x}{-e^{2x}} dx$$

$$= \int \log x \ dx$$

$$|P = x \log x - x|$$

$$Q = \int \frac{f_1 X}{W} dx = \int \frac{x e^x e^x \log x}{-e^{2x}} dx$$
$$= -\int x \log x \ dx$$

use integration by parts,  $u = \log x$ , dv = x dx

$$=> du = \frac{dx}{x}, v = \frac{x^2}{2}$$

$$Q = -\int x \log x \, dx = -\left[\frac{x^2}{2} \log x - \int \frac{x^2}{2} \frac{dx}{x}\right]$$

$$= -\frac{x^2}{2} \log x + \int \frac{x}{2} dx$$

$$Q = -\frac{x^2}{2}\log x + \frac{x^2}{4}$$

$$\therefore PI = Pf_1 + Qf_2$$

$$= (x \log x - x)xe^x + \left(-\frac{x^2}{2}\log x + \frac{x^2}{4}\right)e^x$$

$$= x^2e^x \log x - x^2e^x - \frac{x^2e^x}{2}\log x + \frac{x^2e^x}{4}$$

$$PI = \frac{x^2e^x}{2}\log x - \frac{3x^2e^x}{4}$$

$$\therefore soln \ is \ y = CF + PI$$

$$x^2e^x \qquad 3x^2e^x$$

$$y = (Ax + B)e^{x} + \frac{x^{2}e^{x}}{2}\log x - \frac{3x^{2}e^{x}}{4}$$

 $4. Solve (D^2 + 1)y = \csc x \cot x$ 

$$Gn$$
,  $(D^2 + 1)y = \csc x \cot x$   
 $a.e.$  is  $m^2 + 1 = 0 => m = \pm i$   
 $CF = A \cos x + B \sin x$ 

Here 
$$f_1 = \cos x => f_1' = -\sin x$$

$$f_2 = \sin x => f_2' = \cos x$$

$$W = f_1 f_2' - f_2 f_1'$$

$$= \cos^2 x + \sin^2 x = 1$$
Here  $X = \operatorname{cosec} x \cot x$ 
Then  $PI = Pf_1 + Qf_2$ 

$$where  $P = -\int \frac{f_2 X}{W} dx = -\int \frac{\sin x \operatorname{cosec} x \cot x}{1} dx$ 

$$= -\int \cot x \ dx$$

$$P = -\log(\sin x)$$

$$Q = \int \frac{f_1 X}{W} dx = \int \frac{\cos x \operatorname{cosec} x \cot x}{1} dx$$$$

$$= \int \frac{\cos^2 x}{\sin^2 x} \ dx$$

$$= \int \frac{1 - \sin^2 x}{\sin^2 x} dx = \int (\csc^2 x - 1) dx$$

$$Q = -\cot x - x$$

 $PI = (-\log(\sin x))\cos x - (\cot x + x)\sin x$ 

$$\therefore$$
 soln is  $y = CF + PI$ 

$$y = A\cos x + B\sin x \left(-\log(\sin x)\right)\cos x$$
$$-(\cot x + x)\sin x$$

# Differential Equations of First Order

1. Definitions. 2. Practical approach to differential equations. 3. Formation of a differential equation. 4. Solution of a differential equation—Geometrical meaning—5. Equations of the first order and first degree. 6. Variables separable. 7. Homogeneous equations. 8. Equations reducible to homogeneous form. 9. Linear equations. 10. Bernoulli's equation. 11. Exact equations. 12. Equations reducible to exact equations. 13. Equations of the first order and higher degree. 14. Clairut's equation. 15. Objective Type of Questions.

#### 11.1 DEFINITIONS

(1) A differential equation is an equation which involves differential coefficients or differentials.

Thus

$$(i) e^x dx + e^y dy = 0$$

$$(ii) \ \frac{d^2x}{dt^2} + n^2x = 0$$

(iii) 
$$y = x \frac{dy}{dx} + \frac{x}{dy/dx}$$

(iv) 
$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} / \frac{d^2y}{dx^2} = c$$

(v) 
$$\frac{dx}{dt} - wy = a \cos pt$$
,  $\frac{dy}{dt} + wx = a \sin pt$ 

$$(vi) x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u$$

(*vii*) 
$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$
 are all examples of differential equations.

(2) An ordinary differential equation is that in which all the differential coefficients have reference to a single independent variable. Thus the equations (i) to (v) are all ordinary differential equations.

A partial differential equation is that in which there are two or more independent variables and partial differential coefficients with respect to any of them. Thus the equations (vi) and (vii) are partial differential equations.

(3) The order of a differential equation is the order of the highest derivative appearing in it.

The **degree** of a differential equation is the degree of the highest derivative occurring in it, after the equation has been expressed in a form free from radicals and fractions as far as the derivatives are concerned.

Thus, from the examples above,

(i) is of the first order and first degree;

(ii) is of the second order and first degree;

(iii) written as  $y \frac{dy}{dx} = x \left(\frac{dy}{dx}\right)^2 + x$  is clearly of the first order but of second degree;

and (iv) written as  $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=c^2\left(\frac{d^2y}{dx^2}\right)^2$  is of the second order and second degree.

#### 11.2 PRACTICAL APPROACH TO DIFFERENTIAL EQUATIONS

Differential equations arise from many problems in oscillations of mechanical and electrical systems, bending of beams, conduction of heat, velocity of chemical reactions etc., and as such play a very important role in all modern scientific and engineering studies.

The approach of an engineering student to the study of differential equations has got to be practical unlike that of a student of mathematics, who is only interested in solving the differential equations without knowing as to how the differential equations are formed and how their solutions are physically interpreted.

Thus for an applied mathematician, the study of a differential equation consists of three phases:

- (i) formulation of differential equation from the given physical situation, called modelling.
- (ii) solutions of this differential equation, evaluating the arbitrary constants from the given conditions, and
- (iii) physical interpretation of the solution.

#### 11.3 FORMATION OF A DIFFERENTIAL EQUATION

An ordinary differential equation is formed in an attempt to eliminate certain arbitrary constant from a relation in the variables and constants. It will, however, be seen later that the partial differential equations may be formed by the elimination of either arbitrary constants or arbitrary functions. In applied mathematics, every geometrical or physical problem when translated into mathematical symbols gives rise to a differential equation.

**Example 11.1.** Form the differential equation of simple harmonic motion given by  $x = A \cos(nt + \alpha)$ .

**Solution.** To eliminate the constants A and  $\alpha$  differentiating it twice, we have

$$\frac{dx}{dt} = -nA \sin(nt + \alpha) \text{ and } \frac{d^2x}{dt^2} = -n^2A \cos(nt + \alpha) = -n^2x$$

Thus

$$\frac{d^2x}{dt^2} + n^2x = 0$$

is the desired differential equation which states that the acceleration varies as the distance from the origin.

**Example 11.2.** Obtain the differential equation of all circles of radius a and centre (h, k).

(Andhra, 1999)

**Solution.** Such a circle is  $(x - h)^2 + (y - k)^2 = a^2$ 

AND THE SHA

...(i)

where h and k, the coordinates of the centre, and a are the constants.

Differential it twice, we have

$$(x-h+(y-k)\frac{dy}{dx} = 0$$
 and  $(1+(y-k)\frac{d^2y}{dx^2} + (\frac{dy}{dx})^2 = 0$ 

Then

$$y - k = -\frac{1 + (dy/dx)^2}{d^2y/dx^2}$$

and

$$x - h = -(y - k) \frac{dy}{dx} \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]$$

Substituting these in (i) and simplifying, we get  $[1 + (dy/dx)^2]^3 = a^2(d^2y/dx^2)^2$  ...(ii) as the required differential equation

Writing (ii) in the form 
$$\frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = a$$
,

it states that the radius of curvature of a circle at any point is constant.

**Example 11.3.** Obtain the differential equation of the coaxial circles of the system  $x^2 + y^2 + 2ax + c^2 = 0$  where c is a constant and a is a variable. (J.N.T.U., 2003)

**Solution.** We have 
$$x^2 + y^2 + 2ax + c^2 = 0$$
 ...(*i*)

Differentiating w.r.t. x, 2x + 2ydy/dx + 2a = 0

or

$$2a = -2\left(x + y\frac{dy}{dx}\right)$$

Substituting in (i),  $x^2 + y^2 - 2(x + y dy/dx)x + c^2 = 0$ 

or

$$2xy \, dy/dx = y^2 - x^2 + c^2$$

which is the required differential equation.

#### 11.4 (1) SOLUTION OF A DIFFERENTIAL EQUATION

A solution (or integral) of a differential equation is a relation between the variables which satisfies the given differential equation.

$$x = A \cos(nt + \alpha)$$

...(1)

is a solution of

$$\frac{d^2x}{dt^2} + n^2x = 0 \text{ [Example 11.1]}$$
...(2)

The general (or complete) solution of a differential equation is that in which the number of arbitrary constants is equal to the order of the differential equation. Thus (1) is a general solution (2) as the number of arbitrary constants  $(A, \alpha)$  is the same as the order of (2).

A particular solution is that which can be obtained from the general solution by giving particular values to the arbitrary constants.

For example,

$$x = A\cos\left(nt + \pi/4\right)$$

is the particular solution of the equation (2) as it can be derived from the general solution (1) by putting  $\alpha = \pi/4$ .

A differential equation may sometimes have an additional solution which cannot be obtained from the general solution by assigning a particular value to the arbitrary constant. Such a solution is called a singular solution and is not of much engineering interest.

**Linearly independent solution.** Two solutions  $y_1(x)$  and  $y_2(x)$  of the differential equation

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0$$
 ...(3)

are said to be linearly independent if  $c_1y_1 + c_2y_2 = 0$  such that  $c_1 = 0$  and  $c_2 = 0$ 

If  $c_1$  and  $c_2$  are not both zero, then the two solutions  $y_1$  and  $y_2$  are said to be linearly dependent.

If  $y_1(x)$  and  $y_2(x)$  any two solutions of (3), then their linear combination  $c_1y_1 + c_2y_2$  where  $c_1$  and  $c_2$  are constants, is also a solution of (3).

#### Example 11.4. Find the differential equation whose set of independent solutions is [ex, xex].

**Solution.** Let the general solution of the required differential equation be  $y = c_1 e^x + c_2 x e^x$ ...(i)

Differentiating (i) w.r.t. x, we get

$$y_1 = c_1 e^x + c_2 (e^x + x e^x)$$
  
 $y - y_1 = c_2 e^x$  ...(ii)

Again differentiating (ii) w.r.t. x, we obtain

$$y_1 - y_2 = c_2 e^x$$

...(iii)

Subtracting (iii) from (ii), we get

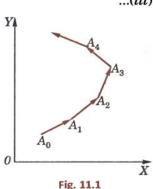
$$y - y_1 - (y_1 - y_2) = 0$$
 or  $y - 2y_1 + y_2 = 0$ 

which is the desired differential equation.

(2) Geometrical meaning of a differential equation. Consider any differential equation of the first order and first degree

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

If P(x, y) be any point, then (1) can be regarded as an equation giving the value of dy/dx (= m) when the values of x and y are known (Fig. 11.1). Let the value of m at the point  $A_0(x_0, y_0)$  derived from (1) be  $m_0$ . Take a neighbouring



point  $A_1(x_1, y_1)$  such that the slope of  $A_0A_1$  is  $m_0$ . Let the corresponding value of m at  $A_1$  be  $m_1$ . Similarly take a neighbouring point  $A_2(x_2, y_2)$  such that the slope of  $A_1A_2$  is  $m_1$  and so on.

If the successive points  $A_0$ ,  $A_1$ ,  $A_2$ ,  $A_3$  ... are chosen very near one another, the broken curve  $A_0A_1A_2A_3$  ... approximates to a smooth curve  $C[y = \phi(x)]$  which is a solution of (1) associated with the initial point  $A_0$   $(x_0, y_0)$ . Clearly the slope of the tangent to C at any point and the coordinates of that point satisfy (1).

A different choice of the initial point will, in general, give a different curve with the same property. The equation of each such curve is thus a **particular solution** of the differential equation (1). The equation of the whole family of such curves is the general solution of (1). The slope of the tangent at any point of each member of this family and the co-ordinates of that point satisfy (1).

Such a simple geometric interpretation of the solutions of a second (or higher) order differential equation is not available.

#### PROBLEMS 11.1

Form the differential equations from the following equations:

1.  $y = ax^3 + bx^2$ .

2.  $y = C_1 \cos 2x + C_2 \sin 2x$ 

(Bhopal, 2008)

3.  $xy = Ae^x + Be^{-x} + x^2$ . (U.P.T.U., 2005)

4.  $y = e^x (A \cos x + B \sin x)$ .

(P.T.U., 2003)

5.  $y = ae^{2x} + be^{-3x} + ce^{x}$ .

#### Find the differential equations of:

6. A family of circles passing through the origin and having centres on the x-axis.

(J.N.T.U., 2006)

- 7. All circles of radius 5, with their centres on the y-axis.
- 8. All parabolas with x-axis as the axis and (a, 0) as focus.
- 9. If  $y_1(x) = \sin 2x$  and  $y_2(x) = \cos 2x$  are two solutions of y'' + 4y = 0, show that  $y_1(x)$  and  $y_2(x)$  are linearly independent solutions.
- 10. Determine the differential equation whose set of independent solutions is  $[e^x, xe^x, x^2e^x]$  (U.P.T.U., 2002)
- 11. Obtain the differential equation of the family of parabolas  $y = x^2 + c$  and sketch those members of the family which pass through (0, 0), (1, 1), (0, 1) and (1, -1) respectively.

#### 11.5 EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE

It is not possible to solve such equations in general. We shall, however, discuss some special methods of solution which are applied to the following types of equations:

- (i) Equations where variables are separable,
- (ii) Homogeneous equations,

(iii) Linear equations,

(iv) Exact equations.

In other cases, the particular solution may be determined numerically (Chapter 31).

#### 11.6 VARIABLES SEPARABLE

If in an equation it is possible to collect all functions of x and dx on one side and all the functions of y and dy on the other side, then the *variables are said to be separable*. Thus the general form of such an equation is f(y)  $dy = \phi(x) dx$ 

Integrating both sides, we get  $\int f(y) dy = \int \phi(x) dx + c$  as its solution.

Example 11.5. Solve 
$$dy/dx = \frac{x(2 \log x + 1)}{\sin y + y \cos y}$$
.

(V.T.U., 2008)

**Solution.** Given equation is  $x (2 \log x + 1) dx = (\sin y + y \cos y) dy$ 

Integrating both sides,  $2 \int (\log x \cdot x + x) dx = \int \sin y dy + \int y \cos y dy + c$ 

$$2\left[\left(\log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} \, dx\right) + \frac{x^2}{2}\right] = -\cos y + \left[y \sin y - \int \sin y \cdot 1 \, dy + c\right]$$

$$\mathbf{or}$$

$$2x^2 \log x - \frac{x^2}{2} + \frac{x^2}{2} = -\cos y + y \sin y + \cos y + c$$

Hence the solution is  $2x^2 \log x - y \sin y = c$ .

**Example 11.6.** Solve 
$$\frac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$$
.

**Solution.** Given equation is 
$$\frac{dy}{dx} = e^{-2y}(e^{3x} + x^2)$$
 or  $e^{2y} dy = (e^{3x} + x^2) dx$ 

Integrating both sides,  $\int e^{2y} dy = \int (e^{3x} + x^2) dx + c$ 

or

$$\frac{e^{2y}}{2} = \frac{e^{3x}}{3} + \frac{x^3}{3} + c \quad \text{or} \quad 3e^{2y} = 2(e^{3x} + x^3) + 6c.$$

Example 11.7. Solve 
$$\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$$
.

(V.T.U., 2005)

**Solution.** Putting x + y = t so that dy/dx = dt/dx - 1

The given equation becomes  $\frac{dt}{dx} - 1 = \sin t + \cos t$ 

 $dt/dx = 1 + \sin t + \cos t$ 

Integrating both sides, we get  $dx = \int \frac{dt}{1 + \sin t + \cos t} + c$ .

or

or

or

or

$$x = \int \frac{2d\theta}{1 + \sin 2\theta + \cos 2\theta} + c$$
 [Putting  $t = 2\theta$ ]
$$= \int \frac{2d\theta}{2\cos^2 \theta + 2\sin \theta \cos \theta} + c = \int \frac{\sec^2 \theta}{1 + \tan \theta} d\theta + c$$

$$= \log (1 + \tan \theta) + c$$

Hence the solution is  $x = \log \left[1 + \tan \frac{1}{2}(x + y)\right] + c$ .

**Example 11.8.** Solve  $dy/dx = (4x + y + 1)^2$ , if y(0) = 1.

**Solution.** Putting 4x + y + 1 = t, we get  $\frac{dy}{dx} = \frac{dt}{dx} - 4$ .

 $\therefore$  the given equation becomes  $\frac{dt}{dx} - 4 = t^2$  or  $\frac{dt}{dx} = 4 + t^2$ 

Integrating both sides, we get  $\int \frac{dt}{4+t^2} = \int dx + c$ 

 $\frac{1}{2} \tan^{-1} \frac{t}{2} = x + c$  or  $\frac{1}{2} \tan^{-1} \left[ \frac{1}{2} (4x + y + 1) \right] = x + c$ .

 $4x + y + 1 = 2 \tan 2(x + c)$ 

When x = 0, y = 1 :  $\frac{1}{2} \tan^{-1}(1) = c$  i.e.  $c = \pi/8$ .

Hence the solution is  $4x + y + 1 = 2 \tan (2x + \pi/4)$ .

**Example 11.9.** Solve  $\frac{y}{x} \frac{dy}{dx} + \frac{x^2 + y^2 - 1}{2(x^2 + y^2) + 1} = 0$ .

(V.T.U., 2003)

**Solution.** Putting  $x^2 + y^2 = t$ , we get  $2x + 2y \frac{dy}{dx} = \frac{dt}{dx}$  or  $\frac{y}{x} \frac{dy}{dx} = \frac{1}{2x} \frac{dt}{dx} - 1$ .

Therefore the given equation becomes  $\frac{1}{2r} \frac{dt}{dr} - 1 + \frac{t-1}{2t+1} = 0$ 

or

$$\frac{1}{2x}\frac{dt}{dx} = 1 - \frac{t-1}{2t+1} = \frac{t+2}{2t+1} \quad \text{or} \quad 2x \, dx = \frac{2t+1}{t+2} dt$$

or

or

$$2x\,dx = \left(2 - \frac{3}{t+2}\right)\,dt$$

Integrating, we get

$$x^{2} = 2t - 3 \log (t + 2) + c$$
  
$$x^{2} + 2y^{2} - 3 \log (x^{2} + y^{2} + 2) + c = 0$$

 $[\because t = x^2 + v^2]$ 

which is the required solution.

#### PROBLEMS 11.2

Solve the following differential equations:

1. 
$$y\sqrt{(1-x^2)} dy + x\sqrt{(1-y^2)} dx = 0$$
.

3. 
$$\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$$
. (P.T.U., 2003)

5. 
$$e^x \tan y \, dx + (1 - e^x) \sec^2 y \, dy = 0$$
. (V.T.U., 2009)

7. 
$$x \frac{dy}{dx} + \cot y = 0$$
 if  $y = \pi/4$  when  $x = \sqrt{2}$ .

9. 
$$\frac{dy}{dx} = e^{2x-3y} + 4x^2 e^{-3y}$$
.

11. 
$$(x+1)\frac{dy}{dx} + 1 = 2e^{-y}$$
. (Madras, 2000 S)

13. 
$$(x + y + 1)^2 \frac{dy}{dx} = 1$$
. (Kurukshetra, 2005)

**15.** 
$$\frac{dy}{dx} = \cos(x + y + 1)$$
 (V.T.U., 2003)

17. 
$$x^4 \frac{dy}{dx} + x^3 y + \csc(xy) = 0$$
.

2. 
$$(x^2 - yx^2) \frac{dy}{dx} + y^2 + xy^2 = 0$$
.

3. 
$$\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$$
. (P.T.U., 2003) 4.  $\frac{y}{x} \frac{dy}{dx} = \sqrt{(1 + x^2 + y^2 + x^2 y^2)}$ . (V.T.U., 2011)

**6.** 
$$\frac{dy}{dx} = xe^{y-x^2}$$
, if  $y = 0$  when  $x = 0$ . (J.N.T.U., 2006)

8. 
$$(xy^2 + x) dx + (yx^2 + y) dy = 0$$
.

$$10. \ y - x \frac{dy}{dx} = a \left( y^2 + \frac{dy}{dx} \right).$$

12. 
$$(x-y)^2 \frac{dy}{dx} = a^2$$
.

14. 
$$\sin^{-1}(dy/dx) = x + y$$

16. 
$$\frac{dy}{dx} - x \tan(y - x) = 1$$
.

#### 11.7 HOMOGENEOUS EQUATIONS

are of the form  $\frac{dy}{dx} = \frac{f(x, y)}{\phi(x, y)}$ 

where f(x, y) and  $\phi(x, y)$  are homogeneous functions of the same degree in x and y (see page 205).

To solve a homogeneous equation (i) Put y = vx, then  $\frac{dy}{dx} = v + x \frac{dv}{dx}$ ,

(ii) Separate the variables v and x, and integrate.

**Example 11.10.** Solve  $(x^2 - y^2) dx - xy dy = 0$ .

**Solution.** Given equation is  $\frac{dy}{dx} = \frac{x^2 - y^2}{xy}$  which is homogeneous in x and y. ...(i)

Put 
$$y = vx$$
, then  $\frac{dy}{dx} = v + x \frac{dv}{dx}$ .  $\therefore$  (i) becomes  $v + x \frac{dv}{dx} = \frac{1 - v^2}{v}$ 

or

$$x\frac{dv}{dx} = \frac{1-v^2}{v} - v = \frac{1-2v^2}{v}$$
.

or

or

or

or

Separating the variables,  $\frac{v}{1-2v^2} dv = \frac{dx}{x}$ 

Integrating both sides,  $\int \frac{v \, dv}{1 - 2v^2} = \int \frac{dx}{x} + c$ 

$$-\frac{1}{4} \int \frac{-4v}{1-2v^2} dv = \int \frac{dx}{x} + c \quad \text{or} \quad -\frac{1}{4} \log (1-2v^2) = \log x + c$$

$$4 \log x + \log (1 - 2v^2) = -4c \quad \text{or} \quad \log x^4 (1 - 2v^2) = -4c$$

$$x^4 (1 - 2v^2/x^2) = e^{-3c} = c'$$
[Put  $v = y/x$ ]

Hence the required solution is  $x^2(x^2 - 2y^2) = c'$ .

**Example 11.11.** Solve 
$$(x \tan y/x - y \sec^2 y/x) dx - x \sec^2 y/x dy = 0.$$
 (V.T.U., 2006)

Solution. The given equation may be rewritten as

$$\frac{dy}{dx} = \left(\frac{y}{x}\sec^2\frac{y}{x} - \tan\frac{y}{x}\right) \cos^2 y/x \qquad ...(i)$$

which is a homogeneous equation. Putting y = vx, (i) becomes  $v + x \frac{dv}{dx} = (v \sec^2 v - \tan v) \cos^2 v$ 

or  $x\frac{dv}{dx} = v - \tan v \cos^2 v - v$ 

Separating the variables  $\frac{\sec^2 v}{\tan v} dv = -\frac{dx}{x}$ 

Integrating both sides  $\log \tan v = -\log x + \log c$ 

 $x \tan v = c$  or  $x \tan y/x = c$ .

**Example 11.12.** Solve  $(1 + e^{x/y}) dx + e^{x/y} (1 - x/y) dy = 0$ .

(P.T.U., 2006; Rajasthan, 2005; V.T.U., 2003)

Solution. The given equation may be rewritten as

$$\frac{dx}{dy} = -\frac{e^{x/y}(1 - x/y)}{1 + e^{x/y}}$$
...(i)

which is a homogeneous equation. Putting x = vy so that (i) becomes

$$v + y \frac{dv}{dy} = -\frac{e^{v}(1-v)}{1+e^{v}}$$
 or  $y \frac{dv}{dy} = -\frac{e^{v}(1-v)}{1+e^{v}} - v = -\frac{v+e^{v}}{1+e^{v}}$ 

Separating the variables, we get

$$-\frac{dy}{y} = \frac{1 + e^{v}}{v + e^{v}} dv = \frac{d(v + e^{v})}{v + e^{v}}$$

Integrating both sides,  $-\log y = \log (v + e^v) + c$ 

$$y(v + e^{v}) = e^{-c}$$
 or  $x + ye^{x/y} = c'$  (say)

which is the required solution.

#### PROBLEMS 11.3

Solve the following differential equations:

1. 
$$(x^2 - y^2) dx = 2xy dy$$
 2.  $(x^2y - 2xy^2) dx - (x^3 - 3x^2y) dy = 0$ . (Bhopal, 2008)

3. 
$$x^2y dx - (x^3 + y^3) dy = 0$$
. (V.T.U., 2010) 4.  $y dx - x dy = \sqrt{x^2 + y^2} dx$ . (Raipur, 2005)

5. 
$$y^2 + x^2 \frac{dy}{dx} = xy \frac{dy}{dx}$$
. 6.  $(3xy - 2ay^2) dx + (x^2 - 2axy) dy = 0$ . (S.V.T.U., 2009)

[Equations solvable like homogeneous equations: When a differential equation contains y/x a number of times, solve it like a homogeneous equation by putting y/x = v].

7. 
$$\frac{dy}{dx} = \frac{y}{x} + \sin\frac{y}{x}$$
. (V.T.U., 2000 S) 8.  $ye^{x/y} dx = (xe^{x/y} + y^2) dy$ . (V.T.U., 2006) 9.  $xy (\log x/y) dx + [y^2 - x^2 \log (x/y)] dy = 0$ . 10.  $x dx + \sin^2(y/x) (ydx - xdy) = 0$ .

9. 
$$xy (\log x/y) dx + [y^2 - x^2 \log (x/y)] dy = 0.$$

10. 
$$x dx + \sin^2(y/x)(ydx - xdy) = 0$$

11. 
$$x \cos \frac{y}{x} (ydx + xdy) = y \sin \frac{y}{x} (xdy - ydx)$$
.

#### 11.8 EQUATIONS REDUCIBLE TO HOMOGENEOUS FORM

The equations of the form 
$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$$
 ...(1)

can be reduced to the homogeneous form as follows:

Case I. When  $\frac{a}{a'} \neq \frac{b}{b'}$ 

Putting

x = X + h, y = Y + k, (h, k being constants)

so that

$$\frac{dX}{dX} = \frac{aX + bY + (ah + bk + c)}{a'X + b'Y + (a'h + b'k + c')}$$
...(2)

Choose h, k so that (2) may become homogeneous.

Put

$$ah + bk + c = 0$$
, and  $a'h + b'k + c' = 0$ 

so that

$$\frac{h}{bc' - b'c} = \frac{k}{ca' - c'a} = \frac{1}{ab' - ba'}$$

$$h = \frac{bc' - b'c}{ab' - b'a}, k = \frac{ca' - c'a}{ab' - ba'}$$
...(3)

or

Thus when  $ab' - ba' \neq 0$ , (2) becomes  $\frac{dY}{dX} = \frac{aX + bY}{a'X + b'Y}$  which is homogeneous in X, Y and can be solved by putting Y = vX.

Case II. When  $\frac{a}{a'} = \frac{b}{b'}$ .

i.e., ab' - b'a = 0, the above method fails as h and k become infinite or indeterminate.

Now 
$$\frac{a}{a'} = \frac{b}{b'} = \frac{1}{m} \text{ (say)}$$

$$\therefore \qquad a' = am, b' = bm \text{ and (1) becomes}$$

$$\frac{dy}{dx} = \frac{(ax + by) + c}{m(ax + by) + c'}$$
...(4)

Put ax + by = t, so that  $a + b \frac{dy}{dx} = \frac{dt}{dx}$ 

or

$$\frac{dy}{dx} = \frac{1}{b} \left( \frac{dt}{dx} - a \right) \quad \therefore \quad \text{(4) becomes } \frac{1}{b} \left( \frac{dt}{dx} - a \right) = \frac{t + c}{mt + c'}$$

or

$$\frac{dt}{dx} = a + \frac{bt + bc}{mt + c'} = \frac{(am + b)t + ac' + bc}{mt + c'}$$

so that the variables are separable. In this solution, putting t = ax + by, we get the required solution of (1).

Example 11.13. Solve 
$$\frac{dy}{dx} = \frac{y + x - 2}{y - x - 4}$$
. (Raipur, 2005)

**Solution.** Given equation is 
$$\frac{dy}{dx} = \frac{y+x-2}{y-x-4}$$
  $\left[ \text{Case } \frac{a}{a'} \neq \frac{b}{b'} \right]$  ...(i)

 $\mathbf{or}$ 

or

or

or

or

Putting x = X + h, y = Y + k, (h, k) being constants) so that dx = dX, dy = dY, (i) becomes

$$\frac{dY}{dX} = \frac{Y + X + (k + h - 2)}{Y - X + (k - h - 4)} \qquad ...(ii)$$

Put k + h - 2 = 0 and k - h - 4 = 0 so that h = -1, k = 3.

$$\therefore (ii) \text{ becomes } \frac{dY}{dX} = \frac{Y+X}{Y-X} \text{ which is homogeneous in } X \text{ and } Y.$$
 ...(iii)

$$\therefore$$
 put  $Y = vX$ , then  $\frac{dY}{dX} = v + X \frac{dv}{dX}$ 

$$\therefore (iii) \text{ becomes} \qquad v + X \frac{dv}{dX} = \frac{v+1}{v-1} \quad \text{or} \quad X \frac{dv}{dX} = \frac{v+1}{v-1} - v = \frac{1+2v-v^2}{v-1}$$

$$\frac{v-1}{1+2v-v^2}\,dv=\frac{dX}{X}.$$

Integrating both sides,  $-\frac{1}{2}\int \frac{2-2v}{1+2v-v^2} dv = \int \frac{dX}{X} + c$ .

or 
$$-\frac{1}{2}\log(1+2v-v^2) = \log X + c$$

or 
$$\log\left(1 + \frac{2Y}{X} - \frac{Y^2}{X^2}\right) + \log X^2 = -2c$$

 $\log (X^2 + 2XY - Y^2) = -2c$  or  $X^2 + 2XY - Y^2 = e^{-2c} = c'$ ...(iv)

Putting X = x - h = x + 1, Y = y - k = y - 3, (*iv*) becomes

$$(x + 1)^2 + 2(x + 1)(y - 3) - (y - 3)^2 = c'$$

 $x^2 + 2xy - y^2 - 4x + 8y - 14 = c'$  which is the required solution. or

**Example 11.14.** Solve (3y + 2x + 4) dx - (4x + 6y + 5) dy = 0.

(Madras, 2000 S)

**Solution.** Given equation is 
$$\frac{dy}{dx} = \frac{(2x+3y)+4}{2(2x+3y)+5}$$
 ...(i)

Putting 2x + 3y = t so that  $2 + 3\frac{dy}{dx} = \frac{dt}{dx}$  : (i) becomes  $\frac{1}{3}\left(\frac{dt}{dx} - 2\right) = \frac{t+4}{2t+5}$ 

$$\frac{dt}{dx} = 2 + \frac{3t+12}{2t+5} = \frac{7t+22}{2t+5}$$
 or  $\frac{2t+5}{7t+22}dt = dx$ 

Integrating both sides,  $\int \frac{2t+5}{7t+22} dt = \int dx + c$ 

$$\int \left(\frac{2}{7} - \frac{9}{7} \cdot \frac{1}{7t + 22}\right) dt = x + c \quad \text{or} \quad \frac{2}{7}t - \frac{9}{49}\log(7t + 22) = x + c$$

Putting t = 2x + 3y, we have  $14(2x + 3y) - 9 \log (14x + 21y + 22) = 49x + 49c$ 

 $21x - 42y + 9 \log (14x + 21y + 22) = c'$  which is the required solution.

#### PROBLEMS 11.4

5.  $\frac{dy}{dx} = \frac{x+y+1}{2x+2y+3}$ 

Solve the following differential equations:

1. 
$$(x-y-2) dx + (x-2y-3) dy = 0$$
.

2. 
$$(2x + y - 3) dy = (x + 2y - 3) dx$$
.

3. 
$$(2x + 5y + 1) dx - (5x + 2y - 1) dy = 0$$
.

4. 
$$\frac{dy}{dx} + \frac{ax + hy + g}{hx + by + f} = 0.$$

6. 
$$(4x-6y-1) dx + (3y-2x-2) dy = 0$$
.

7. 
$$(x + 2y)(dx - dy) = dx + dy$$
.

(Rajasthan, 2006)

(V.T.U., 2009 S; Madras, 2000)

(J.N.T.U., 2000)

(Bhopal, 2002 S; V.T.U., 2001)

#### 11.9 LINEAR EQUATIONS

A differential equation is said to be linear if the dependent variable and its differential coefficients occur only in the first degree and not multiplied together.

Thus the standard form of a linear equation of the first order, commonly known as Leibnitz's linear equation,\* is

$$\frac{dy}{dx}$$
 + Py = Q where, P, Q are the functions of x. ...(1)

To solve the equation, multiply both sides by  $e^{\int Pdx}$  so that we get

$$\frac{dy}{dx} \cdot e^{\int Pdx} + y \cdot (e^{\int Pdx}P) = Qe^{\int Pdx}$$
 i.e.,  $\frac{d}{dx} \cdot (ye^{\int Pdx}) = Qe^{\int Pdx}$ 

Integrating both sides, we get  $ye^{\int Pdx} = \int Qe^{\int Pdx} dx + c$  as the required solution.

Obs. The factor  $e^{\int Pdx}$  on multiplying by which the left-hand side of (1) becomes the differential coefficient of a single function, is called the integrating factor (I.F.) of the linear equation (1).

It is important to remember that I.F. =  $e^{\int Pdx}$  and the solution is y (I.F.) =  $\int Q(I.F.) dx + c$ .

**Example 11.15.** Solve 
$$(x + 1) \frac{dy}{dx} - y e^{3x} (x + 1)^2$$
.

**Solution.** Dividing throughout by (x + 1), given equation becomes

$$\frac{dy}{dx} - \frac{y}{x+1} = e^{3x} (x+1) \text{ which is Leibnitz's equation.} \qquad ...(i)$$

Here

$$P = -\frac{1}{x+1}$$
 and  $\int Pdx = -\int \frac{dx}{x+1} = -\log(x+1) = \log(x+1)^{-1}$ 

:. I.F. = 
$$e^{\int Pdx} = e^{\log(x+1)^{-1}} = \frac{1}{x+1}$$

Thus the solution of (1) is  $y(I.F.) = \int [e^{3x} (x+1)](I.F.) dx + c$ 

 $\mathbf{or}$ 

$$\frac{y}{x+1} = \int e^{3x} dx + c = \frac{1}{3}e^{3x} + c \quad \text{or} \quad y = \left(\frac{1}{3}e^{3x} + c\right)(x+1).$$

Example 11.16. Solve 
$$\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right) \frac{dx}{dy} = 1$$
.

**Solution.** Given equation can be written as 
$$\frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$
 ...(i)

$$I.F. = e^{\int x^{1/2} dx} = e^{2\sqrt{x}}$$

Thus solution of (i) is y (I.F.) =  $\int \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$  (I.F.) dx + c

or 
$$ye^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \cdot e^{2\sqrt{x}} dx + c$$

or 
$$ye^{2\sqrt{x}} = \int x^{-1/2} dx + c$$
 or  $ye^{2\sqrt{x}} = 2\sqrt{x} + c$ .

<sup>\*</sup> See footnote p. 139.

**Example 11.17.** Solve 
$$3x(1-x^2)y^2 \frac{dy}{dx} + (2x^2-1)y^3 = ax^3$$

(Rajasthan, 2006)

**Solution.** Putting  $y^3 = z$  and  $3y^2 \frac{dy}{dx} = \frac{dz}{dx}$ , the given equation becomes

$$x(1-x^2)\frac{dz}{dx} + (2x^2-1)z = ax^3$$
, or  $\frac{dz}{dx} + \frac{2x^2-1}{x-x^3}z = \frac{ax^3}{x-x^3}$  ...(i)

which is Leibnitz's equation in z

$$\therefore \qquad \text{I.F.} = \exp\left(\int \frac{2x^2 - 1}{x - x^3} dx\right)$$

Now 
$$\int \frac{2x^2 - 1}{x - x^3} dx = \int \left( -\frac{1}{x} - \frac{1}{2} \frac{1}{1 + x} + \frac{1}{2} \cdot \frac{1}{1 - x} \right) dx = -\log x - \frac{1}{2} \log (1 + x) - \frac{1}{2} \log (1 - x)$$
$$= -\log \left[ x \sqrt{(1 - x^2)} \right]$$

:. I.F. = 
$$e^{-\log [x\sqrt{(1-x^2)}]} = [x\sqrt{(1-x^2)}]^{-1}$$

Thus the solution of (i) is

$$z(\text{I.F.}) = \int \frac{ax^3}{x - x^3} \text{ (I.F.) } dx + c$$

$$\frac{z}{[x\sqrt{(1 - x^2)}]} = a \int \frac{x^3}{x(1 - x^2)} \cdot \frac{1}{x\sqrt{(1 - x^2)}} dx + c = a \int x(1 - x^2)^{-3/2} dx$$

$$= -\frac{a}{2} \int (-2x)(1 - x^2)^{-3/2} dx + c = a (1 - x^2)^{-1/2} + c$$

or

Hence the solution of the given equation is

$$y^3 = ax + cx \sqrt{(1-x^2)}$$
. [:  $z = y^3$ ]

**Example 11.18.** Solve y (log y) dx + (x - log y) dy = 0.

(U.P.T.U., 2000)

**Solution.** We have 
$$\frac{dx}{dy} + \frac{x}{y \log y} = \frac{1}{y}$$
 ... (i)

which is a Leibnitz's equation in x

$$\therefore \qquad \text{I.F.} = e^{\int \frac{1}{y \log y} dy} = e^{\log (\log y)} = \log y$$

Thus the solution of (i) is x (I.F.) =  $\int \frac{1}{y}$  (I.F.) dy + c

$$x \log y = \int \frac{1}{y} \log y \, dy + c = \frac{1}{2} (\log y)^2 + c$$

i.e.,

$$x = \frac{1}{2} \log y + c (\log y)^{-1}.$$

**Example 11.19.** Solve  $(1 + y^2) dx = (tan^{-1} y - x) dy$ . (Bhopal, 2008; V.T.U., 2008; U.P.T.U., 2005)

**Solution.** This equation contains  $y^2$  and  $\tan^{-1} y$  and is, therefore, not a linear in y; but since only x occurs, it can be written as

$$(1+y^2) \frac{dx}{dy} = \tan^{-1} y - x$$
 or  $\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$ 

which is a Leibnitz's equation in x.

:. I.F. = 
$$e^{\int Pdy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1}y}$$

Thus the solution is x (I.F.) =  $\int \frac{\tan^{-1} y}{1 + y^2}$  (I.F.) dy + c

$$xe^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2} \cdot e^{\tan^{-1}y} dy + c$$

$$= \int te^t dt + c = t \cdot e^t - \int 1 \cdot e^t dt + c$$

$$= t \cdot e^t - e^t + c = (\tan^{-1}y - 1) e^{\tan^{-1}y} + c$$

$$x = \tan^{-1}y - 1 + ce^{-\tan^{-1}y}.$$
[Put  $\tan^{-1}y = t$ ]
$$\therefore \frac{dy}{1+y^2} = dt$$
(Integrating by parts)

or

or

**Example 11.20.** Solve  $r \sin \theta d\theta + (r^3 - 2r^2 \cos \theta + \cos \theta) dr = 0$ .

Solution. Given equation can be rewritten as

$$\sin\theta \frac{d\theta}{dr} + \frac{1}{r} (1 - 2r^2) \cos\theta = -r^2 \qquad \dots (i)$$

Put  $\cos \theta = y$  so that  $-\sin \theta \, d\theta / dr = dy / dr$ 

Then (i) becomes 
$$-\frac{dy}{dr} + \left(\frac{1}{r} - 2r\right)y = -r^2$$
 or  $\frac{dy}{dr} + \left(2r - \frac{1}{r}\right)y = r^2$ 

which is a Leibnitz's equation  $\therefore$  I.F. =  $e^{\int (2r-1/r)dr} = e^{r^2 - \log r} = \frac{1}{r}e^{r^2}$ 

Thus its solution is  $y\left(\frac{1}{r}e^{r^2}\right) = \int r^2 \cdot e^{r^2} \cdot \frac{1}{r} dr + c$ 

 $ye^{r^2}/r = \frac{1}{2}\int e^{r^2}2r\,dr + c = \frac{1}{2}e^{r^2} + c$ 

or  $2e^{r^2}$ 

 $2e^{r^2}\cos\theta = re^{r^2} + 2cr$  or  $r(1 + 2ce^{-r^2}) = 2\cos\theta$ .

#### PROBLEMS 11.5

Solve the following differential equations:

1. 
$$\cos^2 x \frac{dy}{dx} + y = \tan x$$
. 2.  $x \log x \frac{dy}{dx} + y = \log x^2$ . (V.T.U., 2011)

3.  $2y' \cos x + 4y \sin x = \sin 2x$ , given y = 0 when  $x = \pi/3$ . (V.T.U., 2003)

4. 
$$\cosh x \frac{dy}{dx} + y \sinh x = 2 \cosh^2 x \sinh x$$
. (J.N.T.U., 2003)

5. 
$$(1-x^2) \frac{dy}{dx} - xy = 1$$
 (V.T.U., 2010) 6.  $(1-x^2) \frac{dy}{dx} + 2xy = x \sqrt{(1-x^2)}$  (Nagpur, 2009)

7. 
$$\frac{dy}{dx} = -\frac{x + y \cos x}{1 + \sin x}$$
. 8.  $dr + (2r \cot \theta + \sin 2\theta) d\theta = 0$ . (J.N.T.U., 2003)

9. 
$$\frac{dy}{dx} + 2xy = 2e^{-x^2}$$
 (P.T.U., 2005) 10.  $(x + 2y^3) \frac{dy}{dx} = y$ . (Marathwada, 2008)

11. 
$$\sqrt{(1-y^2)} dx = (\sin^{-1} y - x) dy$$
.  
12.  $ye^y dx = (y^3 + 2xe^y) dy$ .  
13.  $(1+y^2) dx + (x-e^{-\tan^{-1} y}) dy = 0$ . (V.T.U., 2006)  
14.  $e^{-y} \sec^2 y dy = dx + x dy$ .

#### 11.10 BERNOULLI'S EQUATION

The equation  $\frac{dy}{dx} + Py = Qy^n$  ...(1)

where P, Q are functions of x, is reducible to the Leibnitz's linear equation and is usually called the Bernoulli's equation\*.

<sup>\*</sup>Named after the Swiss mathematician *Jacob Bernoulli* (1654–1705) who is known for his basic work in probability and elasticity theory. He was professor at Basel and had amongst his students his youngest brother *Johann Bernoulli* (1667–1748) and his nephew *Niklaus Bernoulli* (1687–1759). Johann is known for his basic contributions to Calculus while Niklaus had profound influence on the development of Infinite series and probability. His son *Daniel Bernoulli* (1700–1782) is known for his contributions to kinetic theory of gases and fluid flow.

or

or

To solve (1), divide both sides by 
$$y^n$$
, so that  $y^{-n} \frac{dy}{dx} + Py^{1-n} = Q$  ...(2)

Put  $y^{1-n} = z$  so that  $(1-n) y^{-n} \frac{dy}{dx} = \frac{dz}{dx}$ .

$$\therefore (2) \text{ becomes} \qquad \frac{1}{1-n} \frac{dz}{dx} + Pz = Q \quad \text{or} \quad \frac{dz}{dx} + P(1-n)z = Q(1-n),$$

which is Leibnitz's linear in z and can be solved easily.

**Example 11.21.** Solve  $x \frac{dy}{dx} + y = x^3y^6$ .

**Solution.** Dividing throughout by 
$$xy^6$$
,  $y^{-6} \frac{dy}{dx} + \frac{y^{-5}}{x} = x^2$  ...(i)

Put  $y^{-5} = z$ , so that  $-5y^{-6} \frac{dy}{dx} = \frac{dz}{dx}$  : (i) becomes  $-\frac{1}{5} \frac{dz}{dx} + \frac{z}{x} = x^2$ 

 $\frac{dz}{dx} - \frac{5}{x}z = -5x^2$  which is Leibnitz's linear in z. ...(ii)

 $I.F. = e^{-\int (5/x)dx} = e^{-5 \log x} = e^{\log x^{-5}} = x^{-5}$ 

:. the solution of (ii) is z (I.F.) =  $\int (-5x^2) (I.F.) dx + c$  or  $zx^{-5} = \int (-5x^2) x^{-5} dx + c$ 

 $y^{-5}x^{-5} = -5$ .  $\frac{x^{-2}}{-2} + c$  $[\because z = y^{-5}]$ 

Dividing throughout by  $y^{-5}x^{-5}$ ,  $1 = (2.5 + cx^2) x^3y^5$  which is the required solution.

**Example 11.22.** Solve  $xy (1 + xy^2) \frac{dy}{dx} = 1$ .

(Nagpur, 2009)

Solution. Rewriting the given equation as

$$\frac{dx}{dy} - yx = y^3x^2$$

and dividing by  $x^2$ , we have

$$x^{-2} \frac{dx}{dy} - yx^{-1} = y^3 \qquad ...(i)$$

Putting  $x^{-1} = z$  so that  $-x^{-2} \frac{dx}{dy} = \frac{dz}{dy}$  (i) becomes

 $\frac{dz}{dv} + yz = -y^3$  which is Leibnitz's linear in z.

Here

I.F. = 
$$e^{\int y dy} = e^{y^2/2}$$

 $\therefore$  the solution is  $z(I.F.) = \int (-y^3)(I.F.) dy + c$ 

 $ze^{y^2/2} = -\int y^2 \cdot e^{\frac{1}{2}y^2} \cdot ydy + c$ so that y dy = dt $=-2\int t \cdot e^t dt + c$ [Integrate by parts]  $= -2 [t \cdot e^t - [1 \cdot e^t dt] + c = -2 [te^t - e^t] + c = (2 - y^2) e^{y^2/2} + c$  $z = (2 - v^2) + ce^{-\frac{1}{2}y^2}$  or  $1/x = (2 - y^2) + ce^{-\frac{1}{2}y^2}$ .

or

 $\mathbf{or}$ 

**Note.** General equation reducible to Leibnitz's linear is  $f'(y) \frac{dy}{dx} + Pf(y) = Q$ ...(A) where P, Q are functions of x. To solve it, put f(y) = z.

Example 11.23. Solve  $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$ . (V.T.U., 2011; Marathwada, 2008; J.N.T.U., 2005)

**Solution.** Dividing throughout by  $\cos^2 y$ ,  $\sec^2 y \frac{dy}{dx} + 2x \frac{\sin y \cos y}{\cos^2 y} = x^3$ 

or

$$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$$
 which is of the form (A) above. ...(i)

$$\therefore$$
 put tan  $y = z$  so that  $\sec^2 y \frac{dy}{dx} = \frac{dz}{dx}$   $\therefore$  (i) becomes  $\frac{dz}{dx} + 2xz = x^3$ .

This is Leibnitz's linear equation in z.  $\therefore$  I.F. =  $e^{\int 2x \, dx} = e^{x^2}$ 

:. the solution is 
$$ze^{x^2} = \int e^{x^2} x^3 dx + c = \frac{1}{2} (x^2 - 1) e^{x^2} + c$$
.

Replacing z by tan y, we get tan  $y = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$  which is the required solution.

**Example 11.24.** Solve  $\frac{dz}{dx} + \left(\frac{z}{x}\right) \log z = \frac{z}{x} (\log z)^2$ .

Solution. Dividing by z, the given equation becomes

$$\frac{1}{z}\frac{dz}{dx} + \frac{1}{x}\log z = \frac{1}{x}(\log z)^2 \qquad \dots (i)$$

Put  $\log z = t$  so that

$$\frac{1}{z}\frac{dz}{dx} = \frac{dt}{dx}$$
. : (i) becomes

$$\frac{dt}{dx} + \frac{t}{x} = \frac{t^2}{x} \quad \text{or} \quad \frac{1}{t^2} \frac{dt}{dx} + \frac{1}{x} \cdot \frac{1}{t} = \frac{1}{x} \qquad \dots (ii)$$

This being Bernoulli's equation, put 1/t = v so that (ii) reduces to

$$-\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x} \quad \text{or} \quad \frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x}$$

This is Leibnitz's linear in v.  $\therefore$  I.F. =  $e^{-\int 1/x \, dx} = 1/x$ 

 $\therefore$  the solution is

$$v \cdot \frac{1}{x} = -\int \frac{1}{x} \cdot \frac{1}{x} dx + c = \frac{1}{x} + c$$

Replacing v by  $1/\log z$ , we get  $(x \log z)^{-1} = x^{-1} + c$  or  $(\log z)^{-1} = 1 + cx$  which is the required solution.

#### PROBLEMS 11.6

Solve the following equations:

1. 
$$\frac{dy}{dx} = y \tan x - y^2 \sec x$$
. (P.T.U., 2005)

2. 
$$r \sin \theta - \cos \theta \frac{dr}{d\theta} = r^2$$
. (V.T.U., 2005)

3. 
$$2xy' = 10x^3y^5 + y$$

4. 
$$(x^3y^2 + xy) dx = dy$$
. (B.P.T.U., 2005)

5. 
$$\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$$
. (Bhillai, 2005)

6. 
$$x(x-y) dy + y^2 dx = 0$$
. (I.S.M., 2001)

7. 
$$\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$$
. (Bhopal, 2009)

8. 
$$e^{y} \left( \frac{dy}{dx} + 1 \right) = e^{x}$$
. (V.T.U., 2009)

9. 
$$\sec^2 y \frac{dy}{dx} + x \tan y = x^3$$
.

10. 
$$\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$$
. (Sambalpur, 2002)

11. 
$$\frac{dy}{dx} = \frac{y}{x - \sqrt{(xy)}}$$
. (V.T.U., 2011)

12. 
$$(y \log x - 2) y dx - x dy = 0.$$
 (V.T.U., 2006)

#### 11.11 EXACT DIFFERENTIAL EQUATIONS

(1) **Def.** A differential equation of the form  $\mathbf{M}(x, y) \, dx + \mathbf{N}(x, y) \, dy = 0$  is said to be **exact** if its left hand member is the exact differential of some function u(x, y) i.e., du = Mdx + Ndy = 0. Its solution, therefore, is u(x, y) = c.

(2) **Theorem.** The necessary and sufficient condition for the differential equation Mdx + Ndy = 0 to be exact is

$$\frac{\partial \mathbf{M}}{\partial \mathbf{y}} = \frac{\partial \mathbf{N}}{\partial \mathbf{x}}$$

Condition is necessary:

The equation Mdx + Ndy = 0 will be exact, if

$$Mdx + Ndy \equiv du \qquad ...(1)$$

where u is some function of x any y.

But  $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \qquad ...(2)$ 

equating coefficients of dx and dy in (1) and (2), we get  $M = \frac{\partial u}{\partial x}$  and  $N = \frac{\partial u}{\partial y}$ 

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$
But
$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \tag{Assumption}$$

 $\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ which is the necessary condition for exactness.}$ 

Condition is sufficient: i.e., if  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ , then Mdx + Ndy = 0 is exact.

Let  $\int Mdx = u$ , where y is supposed constant while performing integration.

Then 
$$\frac{\partial}{\partial x} \left( \int M dx \right) = \frac{\partial u}{\partial x}, \quad i.e., \quad M = \frac{\partial u}{\partial x}$$

$$\therefore \qquad \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{or} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right)$$

$$\text{and} \quad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

Interating both sides w.r.t. x (taking y as constant).

$$N = \frac{\partial u}{\partial y} + f(y)$$
, where  $f(y)$  is a function of y alone. ...(4)

$$\therefore Mdx + Ndy = \frac{\partial u}{\partial x} dx + \left\{ \frac{\partial u}{\partial y} + f(y) \right\} dy$$

$$= \left\{ \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \right\} + f(y) dy = du + f(y) dy = d \left[ u + \int f(y) dy \right] \qquad \dots (5)$$

which shows that Mdx + Ndy = 0 is exact.

(3) Method of solution. By (5), the equation Mdx + Ndy = 0 becomes  $d[u + \int f(y)dy] = 0$ 

Integrating 
$$u + \int f(y) dy = 0$$
.  
But  $u = \int_{y \text{ constant}} M dx$  and  $f(y) = \text{terms of } N \text{ not containing } x$ .

 $\therefore$  The solution of Mdx + Ndy = 0 is

$$\int_{(y \text{ cons.})} \mathbf{M} \, d\mathbf{x} + \int (\text{terms of N not containing } \mathbf{x}) \, d\mathbf{y} = \mathbf{c}$$

provided 
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

**Example 11.25.** Solve  $(y^2 e^{xy^2} + 4x^3) dx + (2xy e^{xy^2} - 3y^2) dy = 0$ .

(V.T.U., 2006)

Solution. Here

$$M = y^2 e^{xy^2} + 4x^3 \quad \text{and} \quad N = 2xy e^{xy^2} - 3y^2$$
$$\frac{\partial M}{\partial y} = 2y e^{xy^2} + y^2 e^{xy^2} \cdot 2xy = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y \text{ const.})} M dx + \int (\text{terms of } N \text{ not containing } x) dy = c$$

i.e.,

$$\int_{(y \text{ const.})} (y^2 e^{y^2 x} + 4x^3) dx + \int (-3y^2) dy = c \quad \text{or} \quad e^{xy^2} + x^4 - y^3 = c.$$

**Example 11.26.** Solve  $\left\{ y \left( 1 + \frac{1}{x} \right) + \cos y \right\} dx + (x + \log x - x \sin y) dy = 0.$ 

(Marathwada, 2008 S; V.T.U., 2006)

**Solution.** Here  $M = y(1 + 1/x) + \cos y$  and  $N = x + \log x - x \sin y$ 

$$\frac{\partial M}{\partial y} = 1 + 1/x - \sin y = \frac{\partial N}{\partial x}$$

Then the equation is exact and its solution is

$$\int_{(y \text{ const})} M dx + \int (\text{terms of } N \text{ not containing } x) dy = c$$

$$\int_{(y \text{ const})} \left\{ \left( 1 + \frac{1}{x} \right) y + \cos y \right\} dx = c \quad \text{or} \quad (x + \log x) y + x \cos y = c.$$

**Example 11.27.** Solve  $(1 + 2xy \cos x^2 - 2xy) dx + (\sin x^2 - x^2) dy = 0$ .

**Solution.** Here  $M = 1 + 2xy \cos x^2 - 2xy$  and  $N = \sin x^2 - x^2$ 

$$\frac{\partial M}{\partial y} = 2x \cos x^2 - 2x = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y \text{ const})} M dx + \int (\text{terms of } N \text{ not containing } x) = c$$

$$\int_{(y \text{ const})} (1 + 2xy \cos x^2 - 2xy) dx = c \quad \text{or} \quad x + y \left[ \int \cos x^2 \cdot 2x dx - \int 2x \, dx \right] = c$$

$$x + y \sin x^2 - yx^2 = c.$$

i.e., or

Example 11.28. Solve 
$$\frac{dy}{dx} + \frac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0.$$

(Kurukshetra, 2005)

Solution. Given equation can be written as

$$(y \cos x + \sin y + y) dx + (\sin x + x \cos y + x) dy = 0.$$

Here  $M = y \cos x + \sin y + y$  and  $N = \sin x + x \cos y + x$ .

$$\therefore \frac{\partial M}{\partial y} = \cos x + \cos y + 1 = \frac{\partial N}{\partial x}.$$

Thus the equation is exact and its solution is

$$\int_{(y \text{ const.})} M dx + \int (\text{terms of } N \text{ not containing } x) dy = c$$

i.e., 
$$\int_{(y \cos x + \sin y + y)} dx + \int_{(0)} dx = c \quad \text{or} \quad y \sin x + (\sin y + y) x = c.$$

or

or

or

or

[By componendo & dividendo]

**Example 11.29.** Solve 
$$(2x^2 + 3y^2 - 7) x dx - (3x^2 + 2y^2 - 8) y dy = 0$$
.

(U.P.T.U., 2005)

Solution. Given equation can be written as

$$\frac{ydy}{xdx} = \frac{2x^2 + 3y^2 - 7}{3x^2 + 2y^2 - 8}$$

$$\frac{ydy + xdx}{ydy - xdx} = \frac{5(x^2 + y^2 - 3)}{-x^2 + y^2 + 1}$$

$$\frac{xdx + ydy}{x^2 + y^2 - 3} = 5 \cdot \frac{xdx - ydy}{x^2 - y^2 - 1}$$

Integrating both sides, we get

$$\int \frac{2xdx + 2ydy}{x^2 + y^2 - 3} = 5 \int \frac{2xdx - 2ydy}{x^2 - y^2 - 1} + c$$

$$\log (x^2 + y^2 - 3) = 5 \log (x^2 - y^2 - 1) + \log c'$$

$$x^2 + y^2 - 3 = c' (x^2 - y^2 - 1)^5$$

[Writing  $c = \log c'$ ]

which is the required solution.

#### **PROBLEMS 11.7**

Solve the following equations:

1. 
$$(x^2 - ay) dx = (ax - y^2) dy$$
.

2. 
$$(x^2 + y^2 - a^2) x dx + (x^2 - y^2 - b^2) y dy = 0$$

(Kurukshetra, 2005)

3. 
$$(x^2 - 4xy - 2y^2) dx + (y^2 - 4xy - 2x^2) dy = 0$$
.

4. 
$$(x^4 - 2xy^2 + y^4) dx - (2x^2y - 4xy^3 + \sin y) dy = 0$$

5. 
$$ye^{xy}dx + (xe^{xy} + 2y) dy = 0$$

6. 
$$(5x^4 + 3x^2y^2 - 2xy^3) dx + (2x^3y - 3x^2y^2 - 5y^4) dy = 0$$

(V.T.U., 2008)

7. 
$$(3x^2 + 6xy^2) dx + (6x^2y + 4y^3) dy = 0$$

8. 
$$\frac{2x}{y^3}dx + \frac{y^2 - 3x^2}{y^4}dy = 0$$

9. 
$$y \sin 2x dx - (1 + y^2 + \cos^2 x) dy = 0$$

10. 
$$(\sec x \tan x \tan y - e^x) dx + \sec x \sec^2 y dy = 0$$

11. 
$$(2xy + y - \tan y) dx + x^2 - x \tan^2 y + \sec^2 y) dy = 0$$
.

(Marathwada, 2008)

(Nagpur, 2009)

#### 11.12 EQUATIONS REDUCIBLE TO EXACT EQUATIONS

Sometimes a differential equation which is not exact, can be made so on multiplication by a suitable factor called an *integrating factor*. The rules for finding integrating factors of the equation  $\dot{M}dx + Ndy = 0$  are as follows:

(1) I.F. found by inspection. In a number of cases, the integrating factor can be found after regrouping the terms of the equation and recognizing each group as being a part of an exact differential. In this connection the following integrable combinations prove quite useful:

$$xdy + ydx = d(xy)$$

$$\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right); \frac{xdy - ydx}{xy} = d\left[\log\left(\frac{y}{x}\right)\right]$$

$$\frac{xdy - ydx}{y^2} = -d\left(\frac{x}{y}\right); \frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$$

$$\frac{xdy - ydx}{x^2 - y^2} = d\left(\frac{1}{2}\log\frac{x + y}{x - y}\right).$$

**Example 11.30.** Solve  $y(2xy + e^x) dx = e^x dy$ .

(Kurukshetra, 2005)

**Solution.** It is easy to note that the terms  $ye^xdx$  and  $e^xdy$  should be put together.

$$(ve^x dx - e^x dv) + 2xv^2 dx = 0$$

Now we observe that the term  $2xy^2 dx$  should not involve  $y^2$ . This suggests that  $1/y^2$  may be I.F. Multiplying throughout by  $1/y^2$ , it follows

$$\frac{ye^x dx - e^x dy}{y^2} + 2xdx = 0 \quad \text{or} \quad d\left(\frac{e^x}{y}\right) + 2xdx = 0$$

Integrating, we get  $\frac{e^x}{y} + x^2 = c$  which is the required solution.

(2) I.F. of a homogeneous equation. If Mdx + Ndy = 0 be a homogeneous equation in x and y, then 1/(Mx + Ny) is an integrating factor  $(Mx + Ny \neq 0)$ .

**Example 11.31.** Solve 
$$(x^2y - 2xy^2) dx - (x^3 - 3x^2y) dy = 0$$
.

(Osmania, 2003 S)

**Solution.** This equation is homogeneous in x and y.

$$\therefore I.F. = \frac{1}{Mx + Ny} = \frac{1}{(x^2y - 2xy^2)x - (x^3 - 3x^2y)y} = \frac{1}{x^2y^2}$$

Multiplying throughout by  $1/x^2y^2$ , the equation becomes

$$\left(\frac{1}{y} - \frac{2}{x}\right) dx - \left(\frac{x}{y^2} - \frac{3}{y}\right) dy = 0$$
 which is exact.

 $\therefore \text{ the solution is } \int_{(y \text{ const})} M dx + \int (\text{terms of } N \text{ not containing } x) \, dy = c \text{ or } \frac{x}{y} - 2 \log x + 3 \log y = c.$ 

(3) I.F. for an equation of the type  $f_1(xy)ydx + f_2(xy)xdy = 0$ .

If the equation Mdx + Ndy = 0 be of this form, then 1/(Mx - Ny) is an integrating factor  $(Mx - Ny \neq 0)$ .

**Example 11.32.** Solve 
$$(1 + xy) ydx + (1 - xy) xdy = 0$$
.

(S.V.T.U., 2008)

**Solution.** The given equation is of the form  $f_1(xy) y dx + f_2(xy) x dy = 0$ 

Here

$$M = (1 + xy) y, N = (1 - xy) x.$$

··

or

I.F. = 
$$\frac{1}{Mx - Ny} = \frac{1}{(1 + xy)yx - (1 - xy)xy} = \frac{1}{2x^2y^2}$$

Multiplying throughout by  $1/2x^2y^2$ , it becomes

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0$$
, which is an exact equation.

:. the solution is  $\int_{(y \text{ const})} M dx + \int (\text{terms of } N \text{ not containing } x) dy = c$ 

$$\frac{1}{2y}\left(-\frac{1}{x}\right) + \frac{1}{2}\log x - \frac{1}{2}\log y = c \quad \text{or} \quad \log \frac{x}{y} - \frac{1}{xy} = c'.$$

(4) In the equation Mdx + Ndy = 0,

(a) if 
$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$$
 be a function of  $x$  only =  $f(x)$  say, then  $e^{\int f(x)dx}$  is an integrating factor.

(b) if 
$$\frac{\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}}{M}$$
 be a function of y only =  $F(y)$  say, then  $e^{\int F(y)dy}$  is an integrating factor.

**Example 11.33.** Solve 
$$(xy^2 - e^{1/x^3}) dx - x^2 y dy = 0$$
.

(S.V.T.U., 2009; Mumbai, 2007)

**Solution.** Here 
$$M = xy^2 - e^{1/x^3}$$
 and  $N = -x^2y$   $\partial M = \partial N$ 

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial V}{\partial x}}{N} = \frac{2xy - (-2xy)}{-x^2y} = -\frac{4}{x}$$
 which is a function of x only.

:. I.F. = 
$$e^{\int \frac{-4}{x} dx} = e^{-4 \log x} = x^{-4}$$

Multiplying throughout by  $x^{-4}$ , we get  $\left(\frac{y^2}{x^3} - \frac{1}{4^4}e^{1/x^3}\right)dx - \frac{y}{x^2}dy = 0$ 

which is an exact equation.

:. the solution is  $\int_{(y \text{ const})} (Mdx) + \int (\text{terms of } N \text{ not containing } x) dy = c$ .

or  $\int \left( \frac{y^2}{x^3} - \frac{1}{x^4} e^{1/x^3} \right) dx + 0 = c$ 

 $-\frac{y^2x^{-2}}{2} + \frac{1}{3}\int e^{x^{-3}} \left(-3x^{-4}\right) dx = c \text{ or } \frac{1}{3}e^{x^{-3}} - \frac{1}{2}\frac{y^2}{x^2} = c.$ 

Otherwise it can be solved as a Bernoulli's equation (§ 11.10)

**Example 11.34.** Solve  $(xy^3 + y) dx + 2 (x^2y^2 + x + y^4) dy = 0$ .

**Solution.** Here  $M = xy^3 + y$ ,  $N = 2(x^2y^2 + x + y^4)$ 

$$\therefore \frac{1}{M} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = \frac{1}{y(xy^2 + 1)} (4xy^2 + 2 - 3xy^2 - 1) = \frac{1}{y}, \text{ which is a function of } y \text{ alone.}$$

$$\therefore \qquad \text{I.F.} = e^{\int 1/y \, dy} = e^{\log y} = y$$

Multiplying throughout by y, it becomes  $(xy^4 + y^2) dx + (2x^2y^3 + 2xy + 2y^5) dy = 0$ , which is an exact equation.

: its solution is  $\int_{(y \text{ const})} (Mdx) + \int (\text{terms of } N \text{ not containing } x) dy = 0$ 

or

or

$$\int_{(y \text{ const})} (xy^4 + y^2) dx + \int 2y^5 dy = c \quad \text{or} \quad \frac{1}{2} x^2 y^4 + xy^2 + \frac{1}{3} y^6 = c.$$

**Example 11.35.** Solve  $(y \log y) dx + (x - \log y) dy = 0$ 

(U.P.T.U., 2004)

**Solution.** Here  $M = y \log y$  and  $N = x - \log y$ 

$$\therefore \frac{1}{M} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = \frac{1}{y \log y} (1 - \log y - 1) = -\frac{1}{y}, \text{ which is a function of } y \text{ alone.}$$

$$\therefore \qquad \text{I.F.} = e^{-\int \frac{1}{y} dy} = e^{-\log y} = \frac{1}{y}$$

Multiplying the given equation throughout by 1/y, it becomes

$$\log y \, dx + \frac{1}{y} (x - \log y) \, dy = 0$$

which is an exact equation

$$\left[\because \frac{\partial}{\partial y} (\log y) = \frac{\partial}{\partial x} \left( \frac{x - \log y}{y} \right) \right]$$

: its solution is 
$$\int_{(y \text{ const})} (Mdx) + \int (\text{terms of } N \text{ not containing } x) dy = c$$

or  $\log y \int dx + \int \left(\frac{\log y}{y}\right) dx$ 

$$\log y \int dx + \int \left(\frac{-\log y}{y}\right) dy = c \qquad \text{or} \quad x \log y - \frac{1}{2} (\log y)^2 = c.$$

(5) For the equation of the type

$$x^ay^b\left(mydx+nxdy\right)+x^{a'}y^{b'}\left(m'ydx+n'xdy\right)=0,$$

an integrating factor is  $x^h y^k$ 

where 
$$\frac{a+h+1}{m} = \frac{b+k+1}{n}, \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}.$$

**Example 11.36.** Solve  $y(xy + 2x^2y^3) dx + x (xy - x^2y^2) dy = 0$ .

(Hissar, 2005; Kurukshetra, 2005)

**Solution.** Rewriting the equation as  $xy (ydx + xdy) + x^2y^2 (2ydx - xdy) = 0$  and comparing with  $x^ay^b (mydx + nxdy) + x^a'y^{b'} (m'ydx + n'xdy) = 0$ ,

we have a = b = 1, m = n = 1; a' = b' = 2, m' = 2, n' = -1.

$$I.F. = x^h y^k.$$

where

$$\frac{a+h+1}{m} = \frac{b+k+1}{n}, \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$$
$$\frac{1+h+1}{1} = \frac{1+k+1}{1}, \frac{2+h+1}{2} = \frac{2+k+1}{-1}$$

or

i.e.

Solving these, we get h = k = -3.  $\therefore$  I.F. =  $1/x^3y^3$ .

Multiplying throughout by  $1/x^3y^3$ , it becomes

$$\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dx = 0$$
, which is an exact equation.

 $\therefore \text{ The solution is } \int_{(y \, const)} M dx + \int (\text{terms of } N \text{ not containing } x) \, dy = c$ 

h-k=0, h+2k+9=0

or

$$\frac{1}{y}\left(-\frac{1}{x}\right) + 2\log x - \log y = c$$
 or  $2\log x - \log y - 1/xy = c$ .

#### PROBLEMS 11.8

Solve the following equations:

1. 
$$xdy - ydx + a(x^2 + y^2) dx = 0$$
.

$$3. \quad ydx - xdy + \log x \, dx = 0.$$

5. 
$$(x^3y^2 + x) dy + (x^2y^3 - y) dx = 0$$
.

7. 
$$(y^4 + 2y) dx + (xy^3 + 2y^4 - 4x) dy = 0$$
.

9. 
$$x^4 \frac{dy}{dx} + x^3y + \csc(xy) = 0$$
.

11. 
$$ydx - xdy + 3x^2y^2e^{x^3} dx = 0$$
. (Kurukshetra, 2006)

13. 
$$2ydx + x(2 \log x - y) dy = 0$$
. (P.T.U., 2005)

2. 
$$xdx + ydy = \frac{a^2(xdy - ydx)}{x^2 + y^2}$$
. (U.P.T.U., 2005)

4. 
$$\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2}$$
.

6. 
$$(x^2y^2 + xy + 1)ydx + (x^2y^2 - xy + 1)xdy = 0$$
.

8. 
$$(4xy + 3y^2 - x) dx + x (x + 2y) dy = 0$$
 (Mumbai, 2006)

10. 
$$(y-xy^2) dx - (x+x^2y) dy = 0$$
 (Mumbai, 2006)

12. 
$$(y^2 + 2x^2y) dx + (2x^3 - xy) dy = 0$$
. (Rajasthan, 2005)

#### 11.13 EQUATIONS OF THE FIRST ORDER AND HIGHER DEGREE

As dy/dx will occur in higher degrees, it is convenient to denote dy/dx by p. Such equations are of the form f(x, y, p) = 0. Three cases arise for discussion :

Case. I. Equation solvable for p. A differential equation of the first order but of the nth degree is of the form

where  $P_1, P_2, ..., P_n$  are functions of x and y.

Splitting up the left hand side of (1) into n linear factors, we have

$$[p-f_1(x,y)][p-f_2(x,y)]...[p-f_n(x,y)]=0.$$

Equating each of the factors to zero,

$$p = f_1(x, y), p = f_2(x, y), ..., p = f_n(x, y)$$

Solving each of these equations of the first order and first degree, we get the solutions

$$F_1(x, y, c) = 0, F_2(x, y, c) = 0, ..., F_n(x, y, c) = 0.$$

or

 $\mathbf{or}$ 

These n solutions constitute the general solution of (1).

Otherwise, the general solution of (1) may be written as

$$F_1(x, y, c) \cdot F_2(x, y, c) \cdot \dots F_n(x, y, c) = 0.$$

**Example 11.37.** Solve 
$$\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$$
.

**Solution.** Given equation is 
$$p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}$$
 where  $p = \frac{dy}{dx}$  or  $p^2 + p\left(\frac{y}{x} - \frac{x}{y}\right) - 1 = 0$ .

Factorising (p + y/x)(p - x/y) = 0.

Thus we have p + y/x = 0 ...(i) and p - x/y = 0 ...(ii)

From (i),  $\frac{dy}{dx} + \frac{y}{x} = 0 \text{ or } xdy + ydx = 0$ 

i.e., d(xy) = 0. Integrating, xy = c.

From (ii),  $\frac{dy}{dx} - \frac{x}{y} = 0 \text{ or } xdx - ydx = 0$ 

Integrating,  $x^2 - y^2 = c$ . Thus xy = c or  $x^2 - y^2 = c$ , constitute the required solution.

Otherwise, combining these into one, the required solution can be written as

$$(xy-c)(x^2-y^2-c)=0$$
.

Example 11.38. Solve  $p^2 + 2py \cot x = y^2$ .

(Bhopal, 2008; Kerala, 2005)

**Solution.** We have  $p^2 + 2py \cot x + (y \cot x)^2 = y^2 + y^2 \cot^2 x$ 

or  $p + y \cot x = \pm y \csc x$ 

i.e.,  $p = y (-\cot x + \csc x) \qquad \dots (i)$ 

or  $p = y (-\cot x - \csc x) \qquad \dots (ii)$ 

From (i),  $\frac{dy}{dx} = y(-\cot x + \csc x)$  or  $\frac{dy}{y} = (\csc x - \cot x) dx$ 

Integrating,  $\log y = \log \tan \frac{x}{2} - \log \sin x + \log c = \log \frac{c \tan x/2}{\sin x}$ 

 $y = \frac{c}{2\cos x^2/2}$  or  $y (1 + \cos x) = c$  ...(iii)

From (ii),  $\frac{dy}{dx} = -y(\cot x + \csc x)$  or  $\frac{dy}{y} = -(\cot x + \csc x) dx$ 

Integrating,  $\log y = -\log \sin x - \log \tan \frac{x}{2} + \log c = \log \frac{c}{\sin x \tan \frac{x}{2}}$ 

 $y = \frac{c}{2\sin^2\frac{x}{2}} \quad \text{or} \quad y(1-\cos x) = c \qquad \dots (iv)$ 

Thus combining (iii) and (iv), the required general solution is  $y(1 \pm \cos x) = c$ .

#### PROBLEMS 11.9

Solve the following equations:

1. 
$$y \left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - x = 0$$
. 2.  $p(p+y) = x(x+y)$ . (V.T.U., 2011) 3.  $y = x[p+\sqrt{(1+p^2)}]$ 

**4.** 
$$xy\left(\frac{dy}{dx}\right)^2 - (x^2 + y^2)\frac{dy}{dx} + xy = 0$$
. **5.**  $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$ . (Madras, 2003)